Excursion sets of smooth Gaussian fields and percolation

M. McAuley
Joint work with Dmitry Beliaev and Stephen Muirhead
Mathematical Institute
University of Oxford

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Outline

Percolation models
The standard example

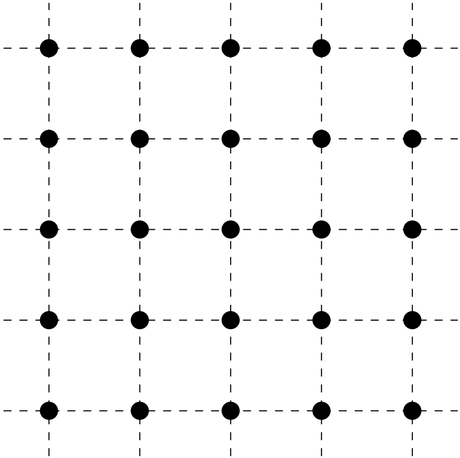

Figure: Bond percolation on the square lattice

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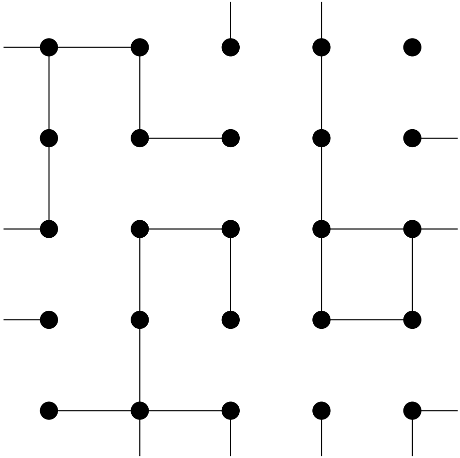

Figure: Bond percolation on the square lattice

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Theorem

For Bernoulli percolation on $\mathbb{Z}^2$ with parameter p, if $p \leq 1/2$ then a.s. there is no infinite open connected component (Harris 1960). If $p > 1/2$ then a.s. there exists a unique infinite open connected component (Kesten 1980).

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Let $C_{[a,b]\times[c,d]}$ be the event that there exists an open path in $[a,b] \times [c,d]$ joining the left and right sides of the rectangle.

Theorem

If $p = 1/2$ then for each $c > 0$ there exists $c_1 > 0$ such that

$$c_1 < \mathbb{P}(C_{[0,R]\times[0,cR]}) < 1 - c_1 \qquad \textit{(RSW)}$$

for all $R > 0$. If $p > 1/2$ then for each $c > 0$ there exists $c_2 > 0$ such that

$$\mathbb{P}(C_{[0,R]\times[0,cR]}) > 1 - e^{-c_2 R} \qquad \textit{(Kesten 1980)}$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a stationary Gaussian field with zero-mean, unit variance and covariance function $\kappa : \mathbb{R}^2 \to [-1, 1]$ and spectral measure ρ, i.e. for $x, y \in \mathbb{R}^2$

$$\kappa(x) = \mathbb{E}(f(y)f(y+x)) = \int_{\mathbb{R}^2} e^{it\cdot x} d\rho(t)$$

We are interested in the geometry of the level sets

$$\mathcal{L}_\ell := \{x \in \mathbb{R}^2 \mid f(x) = \ell\}$$

and (upper) excursion sets

$$\mathcal{E}_\ell := \{x \in \mathbb{R}^2 \mid f(x) \geq \ell\}$$

for $\ell \in \mathbb{R}$.

Gaussian fields

Analogy with percolation models

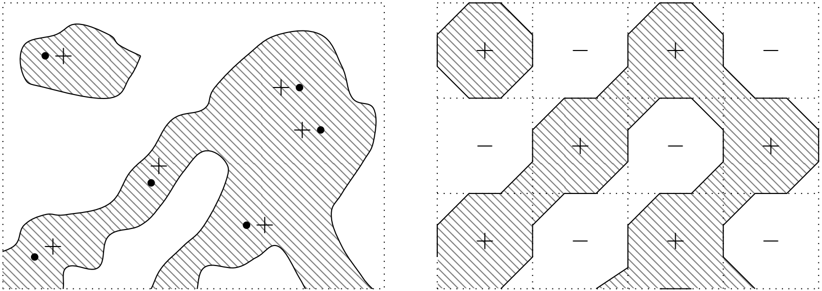
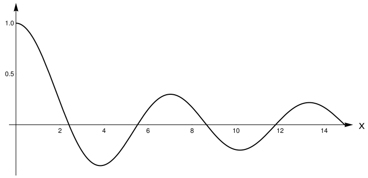
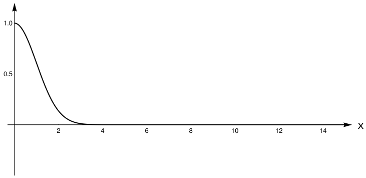


Figure: A Gaussian excursion set $\mathcal{E}_\ell$ and a realisation of a corresponding percolation model with parameter p.

1. Random Plane wave
   - $\kappa(x) = J_0(|x|)$ the zero-th Bessel function
   - Slow decay of correlations $\approx |x|^{-1/2}$
   - Negative correlations
   - Realisations of f are eigenfunctions of the Laplacian with eigenvalue 1
2. Bargmann-Fock field
   - $\kappa(x) = \exp(-|x|^2/2)$
   - Super-exponential decay of correlations
   - $\kappa > 0$ everywhere



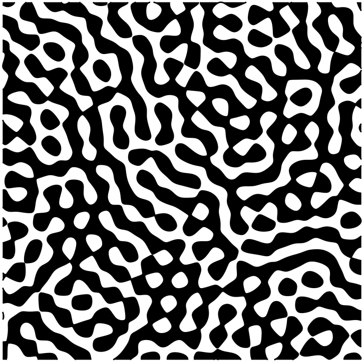
(a) $x \mapsto J_0(x)$



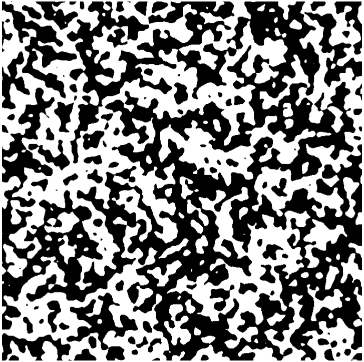
(b) $x \mapsto \exp(-x^2/2)$

Gaussian fields

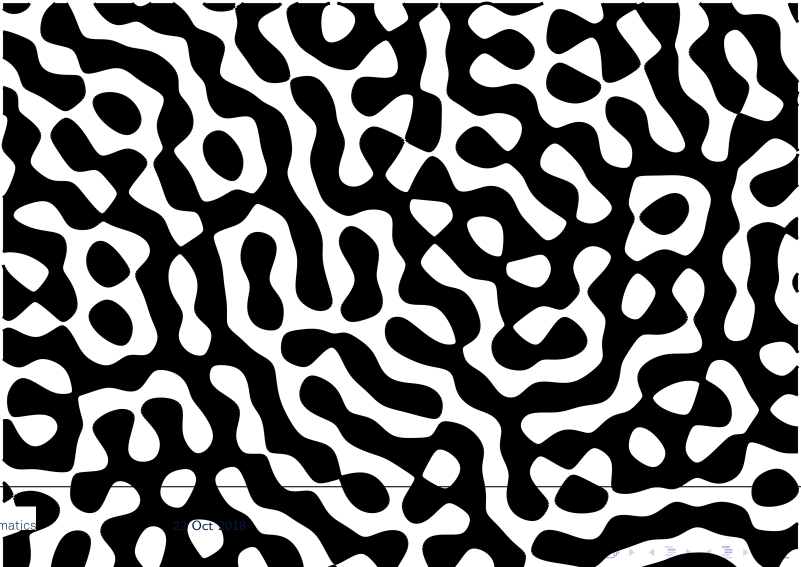
Two important examples



(a) Nodal set of RPW



(b) Nodal set of Bargmann-Fock field



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- (Beffara-Gayet 2016) The Bargmann-Fock field satisfied RSW estimates for $\mathcal{E}_0$ and $\mathcal{L}_0$. The same conclusion holds if $\kappa \geq 0$ and $|\kappa(x)| \lesssim |x|^{-\beta}$ for $\beta > 325$. (Hence $\mathcal{E}_\ell$ contains no infinite component for $\ell \geq 0$).

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- (Rivera-Vanneuville 2018) As above for $\beta > 4$.

Let $C_\ell^{\mathcal{E}}([a,b]\times[c,d])$ be the event that there exists a left-right crossing of $[a,b]\times[c,d]$ in $\mathcal{E}_\ell$ and $C_\ell^{\mathcal{L}}([a,b]\times[c,d])$ the corresponding event for $\mathcal{L}_\ell$.

- (Rivera-Vanneuville 2018) For the Bargmann-Fock field, for each $c>0$, $\ell<0$ there exists $c_1>0$ such that for $R>>0$

$$\mathbb{P}\left(C_\ell^{\mathcal{E}}([0,R]\times[0,cR])\right)>1-e^{-c_1 R}$$

Hence for $\ell<0$, $\mathcal{E}_\ell$ almost surely has a unique unbounded component.

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- (Muirhead-Vanneuville 2018) Suppose $\kappa\geq 0$, $\kappa(x)\lesssim|x|^{2+\epsilon}$ for some $\epsilon>0$ and ρ has a 'nice' density function. Then for each $c>0$, $\ell<0$ there exists $c_1>0$ such that for $R>>0$

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Outline

Conjecture (Bogomolny-Schmit 2001)

The nodal domains of the Random Plane Wave (i.e. components of $\{f \neq 0\}$) can be modelled by critical Bernoulli percolation on the square lattice.
More formally, for $R > 0$ sufficiently large

$$N(R) \approx \mathcal{N}\left(\mu R^2, \sigma^2 R^2\right)$$

where $N(R)$ is the number of components of $\{f \neq 0\}$ in $[0, R]^2$ and μ, σ^2 are explicitly known constants.

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- Numerical results indicate that the prediction for μ is inaccurate (by about 5%).
- However the probability of crossing events for the RPW match those for percolation extremely well numerically.

For $\Omega \subset \mathbb{R}^2$ let $N_{LS}(\ell, \Omega)$ be the number of components of $\{f = \ell\}$ in Ω.

Theorem (Nazarov-Sodin 2016)

If f is ergodic then there exists $c_{NS}(\rho) \geq 0$ such that

$$N_{LS}(0, R \cdot \Omega)/(Area(\Omega)R^2) \to c_{NS}(\rho)$$

a.s. and in L^1.

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Theorem (Kurlberg-Wigman 2018)

If ρ has compact support then there exists $c_{NS}(\rho) \geq 0$ such that

$$\mathbb{E}(N_{LS}(0, [0, R]^2)) = c_{NS}(\rho)\, R^2 + O(R)$$

Moreover $c_{NS}(\rho)$ is continuous in ρ (w.r.t. the w^*-topology).

Number of excursion sets

First moment results

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For $\Omega \subset \mathbb{R}^2$ let $N_{ES}(\ell, \Omega)$ be the number of components of $\{f \geq \ell\}$ in Ω.

Theorem (Nazarov-Sodin 2016)

If f is ergodic then there exists $c_{ES}(\rho, \ell) \geq 0$ such that

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Level sets and excursion sets

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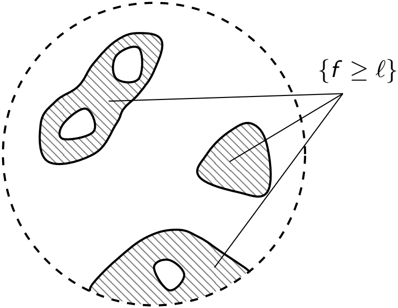

$$\#\{\text{Components of } \{f = \ell\}\} \approx \#\{\text{Components of } \{f \geq \ell\}\} + \#\{\text{Components of } \{f \leq \ell\}\}$$

Corollary

$$c_{NS}(\rho, \ell) = c_{ES}(\rho, \ell) + c_{ES}(\rho, -\ell)$$

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Definition

If f is aperiodic we say that a saddle point x is *lower connected* if it is in the closure of only one component of $\{f < \ell\}$. We say that x is *upper connected* if it is in the closure of only one component of $\{f > \ell\}$.

(When f is periodic, we use a different definition for lower/upper connected saddles.)

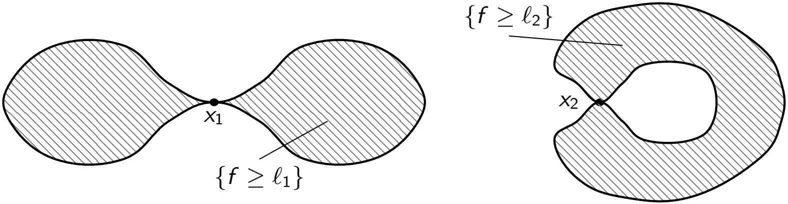

Figure: x_1 is a lower connected saddle and x_2 is an upper connected saddle.

Critical points

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Proposition

Let f satisfy the basic assumptions. There exists a function $p_{s^-} : \mathbb{R} \to [0, \infty)$ such that the following holds. Let $\Omega \subset \mathbb{R}^2$ and let $N_{s^-}[\ell, \infty)$ denote the number of lower connected saddles of f in Ω with level above ℓ. Then

$$\mathbb{E}[N_{s^-}[\ell, \infty)] = \mathrm{Area}(\Omega) \int_{\ell}^{\infty} p_{s^-}(x)\, dx.$$

Analogous statements hold for local maxima, local minima, upper connected saddles and saddles with the densities p_{m^+}, p_{m^-}, p_{s^+} and p_s respectively. These functions can be chosen to satisfy $p_{s^-} + p_{s^+} = p_s$, and such that p_{m^+}, p_{m^-} and p_s are continuous.

Main results

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Theorem

Let f be a Gaussian field satisfying the basic assumptions, and let p_{m^+}, p_{m^-}, p_{s^+}, p_{s^-} denote the critical point densities defined above. Then

$$c_{NS}(\rho,\ell) = \int_\ell^\infty p_{m^+}(x) - p_{s^-}(x) + p_{s^+}(x) - p_{m^-}(x)\,dx \tag{1}$$

$$c_{ES}(\rho,\ell) = \int_\ell^\infty p_{m^+}(x) - p_{s^-}(x)\,dx \tag{2}$$

and hence c_{NS} and c_{ES} are absolutely continuous in ℓ. In addition c_{NS} and c_{ES} are jointly continuous in (ρ,ℓ) provided ρ has a fixed compact support.

Proof: Intuition
Local extrema

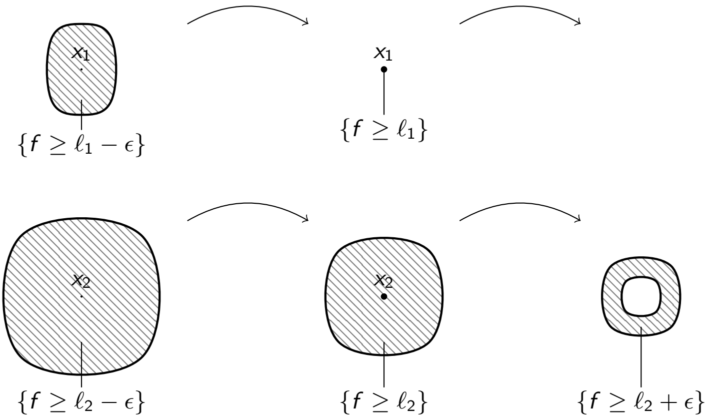

Figure: On raising the level through the local maximum x_1, the number of level set components decreases by one. On passing through the local minimum x_2, the number of level set components increases by one.

Proof: Intuition
Lower connected saddle points

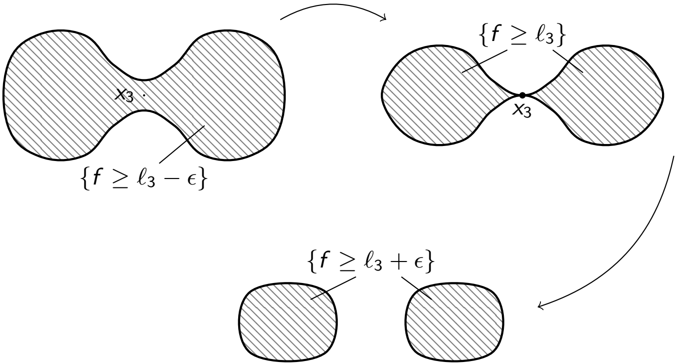

Figure: On raising the level through the lower connected saddle point x_3, the number of level set components increases by one.

Proposition (Cheng-Schwartzman 2017)

Let f be the random plane wave (RPW) so that $\kappa(t) = J_0(|t|)$ (the 0-th Bessel function), then

$$p_{m^+}(x) = p_{m^-}(-x) = \frac{1}{4\sqrt{2}\pi^{3/2}} \left((x^2 - 1)e^{-\frac{x^2}{2}} + e^{-\frac{3x^2}{2}} \right) \mathbb{1}_{x \geq 0}$$

$$p_s(x) = \frac{1}{4\sqrt{2}\pi^{3/2}} e^{-\frac{3x^2}{2}}.$$

Substituting these expressions into the main integral equality and considering the number of 'flip points' (see Kurlberg-Wigman 2018) shows that

Corollary

Let f be the RPW and $\ell \geq 0$, then

$$\frac{1}{4\pi}\ell\,\phi(\ell) \leq c_{ES}(\ell) \leq c_{NS}(\ell) \leq \frac{1}{4\pi}\,\phi(\ell)\left(\sqrt{2}\,\phi(\sqrt{2}\ell) + \ell\left(2\Phi(\sqrt{2}\ell) - 1\right)\right)$$

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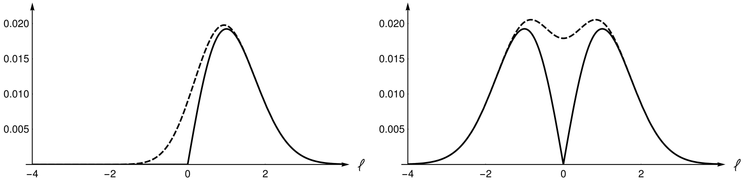

(a) $c_{ES}(\rho, \ell)$ for the RPW　　(b) $c_{NS}(\rho, \ell)$ for the RPW

Figure: Lower bounds (solid) and upper bounds (dashed) for $c_{ES}(\rho, \ell)$ and $c_{NS}(\rho, \ell)$ respectively for the RPW.

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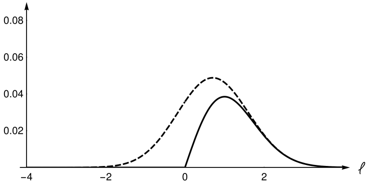

(a) $c_{ES}(\rho, \ell)$ for the Bargmann-Fock field.

(b) $c_{NS}(\rho, \ell)$ for the Bargmann-Fock field.

Figure: Lower bounds (solid) and upper bounds (dashed) for $c_{ES}(\rho, \ell)$ and $c_{NS}(\rho, \ell)$ respectively, where ρ is the spectral measure of the Bargmann-Fock field.

Similar bounds hold for all isotropic fields.

Open questions

1. Can we identify $c_{NS}(\ell)$ or $c_{ES}(\ell)$ for some non-degenerate field and some ℓ? (especially for $\ell = 0$)

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   - We can construct (simple) examples where c_{NS} is differentiable on $\mathbb{R}$ or differentiable on $\mathbb{R}\backslash\{0\}$
   - Partial answer in next section

Outline

1. Percolation-type results for Gaussian fields
2. Number of excursion sets of Gaussian fields: asymptotic mean
3. Number of excursion sets of Gaussian fields: variance

- Based on the Bogomolny-Schmit conjecture, we might expect

$$\mathrm{Var}(N_{LS}(R,\ell)) \simeq R^2$$

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- (Nazarov-Sodin, announced) For random spherical harmonics

$$n^{\sigma} \lesssim \mathrm{Var}(N_{LS}(f_n,0))$$

for some $\sigma > 0$

Theorem

Suppose $\kappa \geq 0$, $\kappa(x) \lesssim |x|^{-(2+\epsilon)}$ *for some* $\epsilon > 0$ *and* ρ *has a 'nice' density function, then* p_{s-}, p_{s+} *can be chosen continuous and so* c_{ES} *and* c_{NS} *are continuously differentiable.*

- These assumptions are used in Muirhead-Vanneuville to prove RSW estimates. These bound the probability of a 'one-arm event', which is used to prove this result. Therefore this theorem could also be proven under other (weaker) conditions which control the probability of 'one-arm events'.

Theorem

Let f be a Gaussian field on $\mathbb{R}^2$ such that $\kappa \geq 0$, $\kappa(x) \lesssim |x|^{-(2+\epsilon)}$ for some $\epsilon > 0$ and ρ has a 'nice' density function. If $c'_{NS}(\ell) \neq 0$ then there exists $c_0(\ell) > 0$ such that

$$\mathrm{Var}(N_{LS}(R,\ell)) \geq c_0(\ell) R^2$$

for all $R > 0$ sufficiently large. The same holds for excursion sets.

Remarks

- $c'_{NS}(0) = 0$ by symmetry, so this result does not apply to nodal sets
- For isotropic fields, $c'_{ES} > 0$ on a neighbourhood of zero and $c'_{ES}(\ell), c'_{NS}(\ell) < 0$ for ℓ large (depending on κ). For example, for the Bargmann-Fock field this holds for $\ell > 1$.

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Proof outline.

Fix a sequence $R_n \to \infty$ and define $X_n := N_{ES}(R_n, \ell)$ and $Y_n := N_{ES}(R_n, \ell + 1/R_n)$.

1. Show that the total variation distance $d_{TV}(X_n, Y_n)$ is small
2. Use differentiability of c_{ES} to show $|\mathbb{E}(X_n - Y_n)| \gtrsim R_n$
3. Use an upper bound on critical points to show $\mathbb{E}((X_n - Y_n)^2) \lesssim R_n^2$
4. By the second moment method, $|X_n - Y_n| \gtrsim R_n$ with probability bounded away from zero
5. A lemma by Chatterjee states that when 1 and 4 hold, X_n has variance of order R_n^2.

□

Extension/open questions

1. Extending differentiability of c_{ES}, c_{NS}
2. Applying lower bound to other levels
3. Upper bound on the variance
4. (Eventually) a central limit theorem