Excursion sets of smooth Gaussian fields and percolation

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Outline

1. Percolation-type results for Gaussian fields

2. Number of excursion sets of Gaussian fields

Percolation models

The standard example

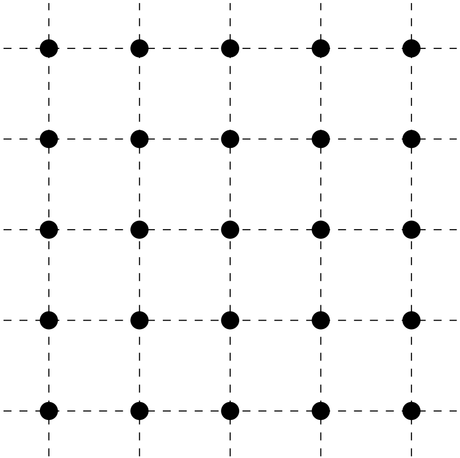

Figure: Bernoulli percolation on the square lattice

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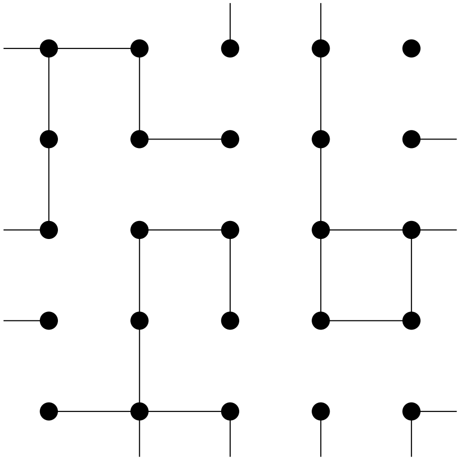

Figure: Bernoulli percolation on the square lattice

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Theorem

For Bernoulli percolation on $\mathbb{Z}^2$ with parameter p, if $p \leq 1/2$ then a.s. there is no infinite open connected component (Harris 1960). If $p > 1/2$ then a.s. there exists a unique infinite open connected component (Kesten 1980).

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Let $C_{[a,b]\times[c,d]}$ be the event that there exists an open path in $[a, b] \times [c, d]$ joining the left and right sides of the rectangle.

Theorem

If $p = 1/2$ then for each $c > 0$ there exists $c_1 > 0$ such that

$$c_1 < \mathbb{P}(C_{[0,R]\times[0,cR]}) < 1 - c_1 \qquad \textit{(RSW)}$$

for all $R > 0$. If $p > 1/2$ then for each $c > 0$ there exists $c_2 > 0$ such that

$$\mathbb{P}(C_{[0,R]\times[0,cR]}) > 1 - e^{-c_2 R} \qquad \textit{(Kesten 1980)}$$

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Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a stationary Gaussian field with zero-mean, unit variance and covariance function $\kappa : \mathbb{R}^2 \to [-1, 1]$ and spectral measure ρ, i.e. for $x, y \in \mathbb{R}^2$

$$\kappa(x) = \mathbb{E}(f(y)f(y + x)) = \int_{\mathbb{R}^2} e^{it \cdot x} d\rho(t)$$

We are interested in the geometry of the level sets

$$\mathcal{L}_\ell := \{x \in \mathbb{R}^2 \mid f(x) = \ell\}$$

and (upper) excursion sets

$$\mathcal{E}_\ell := \{x \in \mathbb{R}^2 \mid f(x) \geq \ell\}$$

for $\ell \in \mathbb{R}$.

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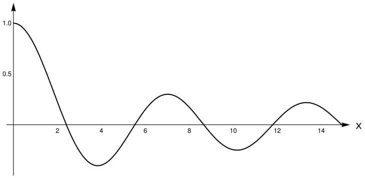
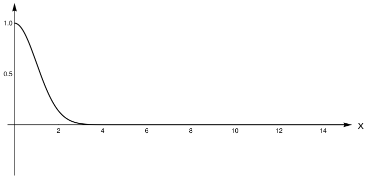
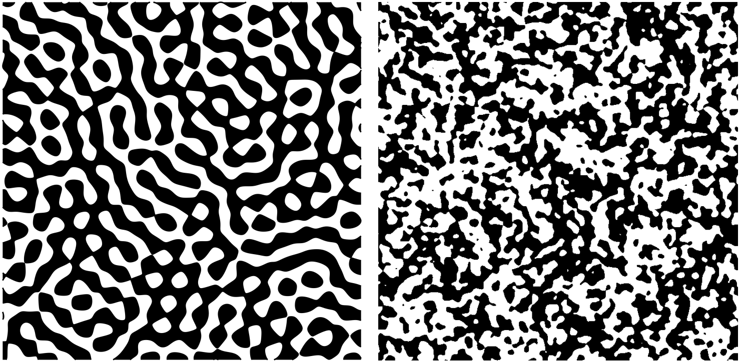
1. Random Plane wave
   - $\kappa(x) = J_0(|x|)$ the zero-th Bessel function
   - Slow decay of correlations $\approx |x|^{-1/2}$
   - Negative correlations
   - Realisations of f are eigenfunctions of the Laplacian with eigenvalue -1
2. Bargmann-Fock field
   - $\kappa(x) = \exp(-|x|^2/2)$
   - Super-exponential decay of correlations
   - $\kappa > 0$ everywhere



(a) $x \mapsto J_0(x)$



(b) $x \mapsto \exp(-x^2/2)$



(a) Nodal set of Random Plane Wave (b) Nodal set of Bargmann-Fock field

Figure: f is positive on black regions and negative on white regions.

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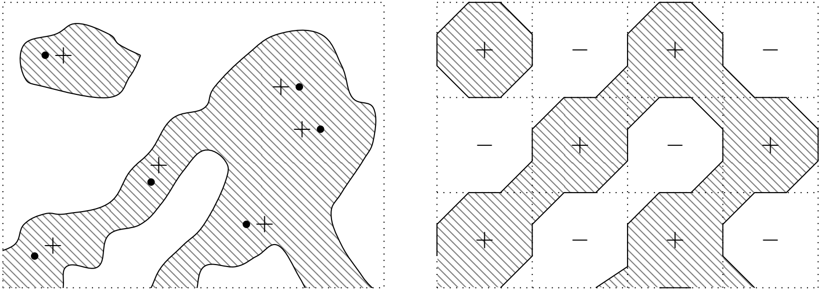

Figure: A Gaussian excursion set $\mathcal{E}_\ell$ and a realisation of a corresponding percolation model with parameter p.

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- (Beffara-Gayet 2016) The Bargmann-Fock field satisfies RSW estimates for $\mathcal{E}_0$ and $\mathcal{L}_0$. The same conclusion holds if $\kappa \geq 0$ and $|\kappa(x)| \lesssim |x|^{-\beta}$ for $\beta > 325$. (Hence $\mathcal{E}_\ell$ contains no infinite component for $\ell \geq 0$).

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- (Rivera-Vanneuville 2018) As above for $\beta > 4$.

Let $C_\ell^{\mathcal{E}}([a,b]\times[c,d])$ be the event that there exists a left-right crossing of $[a,b]\times[c,d]$ in $\mathcal{E}_\ell$ and $C_\ell^{\mathcal{L}}([a,b]\times[c,d])$ the corresponding event for $\mathcal{L}_\ell$.

- (Rivera-Vanneuville 2018) For the Bargmann-Fock field, for each $c>0$, $\ell<0$ there exists $c_1>0$ such that for $R>>0$

$$\mathbb{P}\left(C_\ell^{\mathcal{E}}([0,R]\times[0,cR])\right) > 1-e^{-c_1 R}$$

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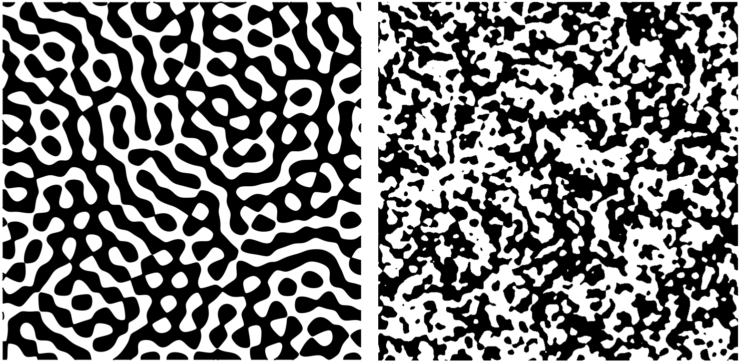
- (Muirhead-Vanneuville 2018) As above for $\kappa\geq 0$, $\kappa(x)\lesssim|x|^{-(2+\epsilon)}$ for some $\epsilon>0$ and ρ that has a density satisfying some technical assumptions.

Outline

Number of excursion sets



(a) Nodal set of Random Plane Wave (b) Nodal set of Bargmann-Fock field

Figure: f is positive on black regions and negative on white regions.

Conjecture (Bogomolny-Schmit 2001)

The nodal domains of the Random Plane Wave (i.e. components of $\{f \neq 0\}$*) can be modelled by critical Bernoulli percolation on the square lattice.*
More formally, for $R > 0$ *sufficiently large*

$$N(R) \approx \mathcal{N}\left(\mu R^2, \sigma^2 R^2\right)$$

where $N(R)$ *is the number of components of* $\{f \neq 0\}$ *in* $[0, R]^2$ *and* μ, σ^2 *are explicitly known constants.*

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- Numerical results indicate that the prediction for μ is inaccurate (by about 5%).
- However the probability of crossing events for the Random Plane Wave match those for percolation extremely well numerically.

Let $N_{LS}(R,\ell)$ be the number of components of $\{f=\ell\}$ in the ball of radius R centred at 0.

Theorem (Nazarov-Sodin 2016)

If f is ergodic then there exists $c_{LS}(\rho) \geq 0$ such that

$$N_{LS}(0,R)/(\pi R^2) \to c_{LS}(\rho)$$

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- (Nazarov-Sodin, announced) For random spherical harmonics

$$n^{\sigma} \lesssim \mathrm{Var}(N_{LS}(f_n,0))$$

for some $\sigma > 0$.

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Theorem

Let f be the Bargmann-Fock field, if $\ell \in (-\epsilon, 0.64) \cup (1.02, \infty)$ *then*

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- Similar bounds hold for other fields with non-negative covariance functions and fast correlation decay.

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Let f be the Random Plane Wave, if $\ell \in (-\infty, 0) \cup (0, 0.87) \cup [1, \infty)$ then

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- As with BF, the same bounds hold for $\ell \neq 0$ such that $c'_{ES}(\ell) \neq 0$ or $c'_{LS}(\ell) \neq 0$.
- This is surprising given the Bogomolny-Schmit conjecture, however our method doesn't apply to $\ell = 0$, so there may be cancellation at this level.

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Lemma (Chatterjee 2017)

Let X_n and Y_n be sequences of random variables and $a_n \in \mathbb{R}$. If $X_n - Y_n$ has fluctuations of order at least a_n and $d_{TV}(X_n - Y_n) \to 0$ then X_n has fluctuations of order at least a_n.

Define $X_R := N_{ES}(R, \ell)$ and $Y_R := N_{ES}(R, \ell + 1/\sqrt{R})$.

Step 1

If $c'_{ES}(\ell) \neq 0$ then

$$\mathbb{E}(X_R - Y_R)| \gtrsim R^{3/2}.$$

It can also be shown that

$$\mathbb{E}((X_R - Y_R)^2) \lesssim R^3.$$

By the second moment method $X_R - Y_R$ has fluctuations of order at least $R^{3/2}$ (and variance of order at least R^3).

Step 2

The Random Plane Wave has an orthogonal expansion

$$f(x) = \sum_{m \in \mathbb{Z}} a_m J_{|m|}(r) e^{im\theta}$$

where (r, θ) represents x in polar coordinates, the a_m are independent standard complex Gaussian random variables and J_m is the m-th Bessel function.
By truncating this expansion and rescaling the a_m we get a bound on $d_{TV}(X_R, Y_R)$, in terms of the total variation distance of two Gaussian random vectors, which tends to zero.

Conclusion

Thank you for listening!