Variance of the number of excursion sets of planar Gaussian fields

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Gaussian Fields: Geometry and Applications in Random Matrix Theory
Queen Mary University of London

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Outline

1. Background and motivation

2. Variance results

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2, stationary Gaussian field with mean zero, variance one, covariance function $\kappa : \mathbb{R}^2 \to [-1, 1]$ and spectral measure ν, i.e. for $x, y \in \mathbb{R}^2$

$$\kappa(x) = \mathbb{E}(f(y)f(y + x)) = \int_{\mathbb{R}^2} e^{it\cdot x} d\nu(t).$$

We are interested in the geometry of the level sets

$$\mathcal{L}_\ell := \left\{x \in \mathbb{R}^2 \mid f(x) = \ell\right\}$$

and (upper) excursion sets

$$\mathcal{E}_\ell := \left\{x \in \mathbb{R}^2 \mid f(x) \geq \ell\right\}$$

for $\ell \in \mathbb{R}$.

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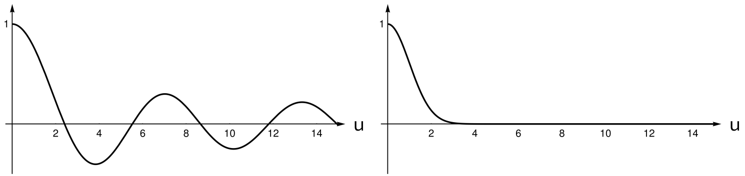
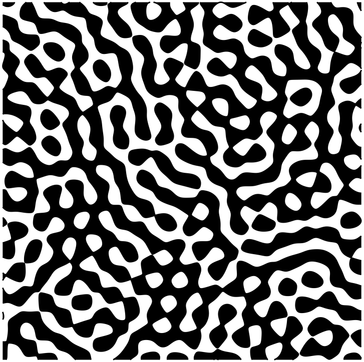
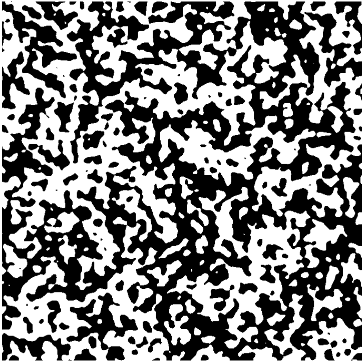
1. Random Plane wave
   - $\kappa(x) = J_0(|x|)$ the zero-th Bessel function
   - ν is normalised Lebesgue measure on the unit circle
   - Realisations of f are eigenfunctions of the Laplacian with eigenvalue -1
2. Bargmann-Fock field
   - $\kappa(x) = \exp(-|x|^2/2)$
   - $\nu(t) = \exp(-|t|^2/2)\, dt$
   - scaling limit of random homogeneous polynomials on $\mathbb{R}P^2$



(a) $u \mapsto J_0(u)$ (b) $u \mapsto \exp(-u^2/2)$



(a) Nodal set (i.e. zero level set) of Random Plane Wave



(b) Nodal set of Bargmann-Fock field

Figure: f is positive on black regions and negative on white regions.

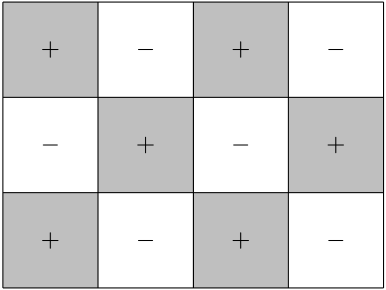


Figure: Approximating nodal lines of the Random Plane Wave by a square grid.

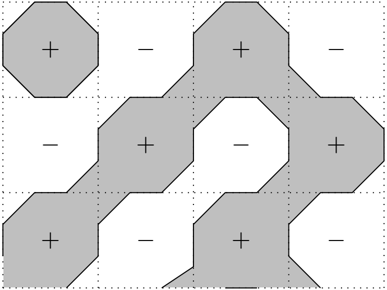


Figure: Adjusting the corners of the grid.

Number of excursion/level sets
The Bogomolny Schmit conjecture

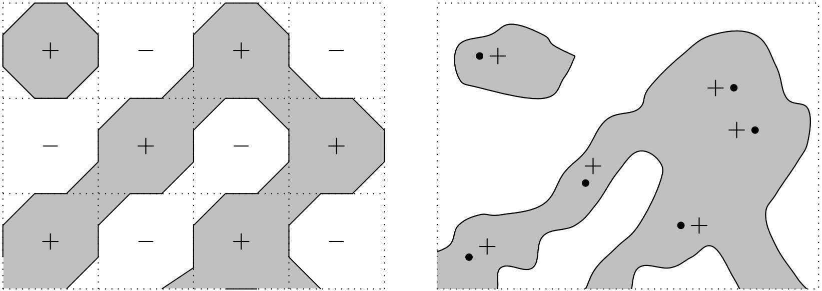


Figure: Adjusting the corners of the grid.

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Conjecture (Bogomolny-Schmit 2001)

The nodal set of the Random Plane Wave (i.e. the set $\{f=0\}$) can be modelled by critical Bernoulli percolation on the square lattice.
In particular, for $R>0$ sufficiently large

$$N(R) \approx \mathcal{N}\left(\mu R^2, \sigma^2 R^2\right)$$

where $N(R)$ is the number of components of $\{f=0\}$ in $[0,R]^2$ and μ, σ^2 are explicitly known constants.

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- They later extend the conjecture to non-zero levels without deriving the analogous mean and variance.
- Numerical results indicate that the prediction for μ is inaccurate (by about 5%).
- However the probability of 'crossing events' for the Random Plane Wave match those for percolation extremely well numerically.

Let $N_{LS}(\ell, R)$ be the number of components of $\{f = \ell\}$ in $[0, R]^2$.

Theorem (Nazarov-Sodin 2016)

If f is ergodic then there exists $c_{LS}(\ell) \geq 0$ such that

$$N_{LS}(\ell, R)/R^2 \to c_{LS}(\ell)$$

a.s. and in L^1 as $R \to \infty$.

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2. Further properties of the functional c_{LS} have been studied (smoothness, lower/upper bounds, monotonicity).

Let $N_{ES}(\ell, R)$ be the number of components of $\{f \geq \ell\}$ in $[0, R]^2$.

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- Based on percolation models, we might expect

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Variance of the number of excursion sets

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$$1 \lesssim \mathrm{Var}(N_{ES}(\ell, R)) \lesssim R^4$$

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- (Nazarov-Sodin, announced) For random spherical harmonics

$$n^{\sigma} \lesssim \mathrm{Var}(N_{LS}(0, f_n))$$

for some $\sigma > 0$.

Theorem

Let f be a Gaussian field such that:

1. $|\partial^\alpha \kappa(x)| \to 0$ *as* $|x| \to \infty$ *for* $|\alpha| \leq 2$
2. ν *has a density* ρ *w.r.t. Lebesgue measure which is 'nice'*
3. $0 < g(r) := \inf_{B(r)} \rho$ *for some* $r > 0$
4. $c'_{ES}(\ell) \neq 0$

then

$$\mathrm{Var}(N_{ES}(\ell, R)) \gtrsim R^2 g(1/R).$$

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- The same conclusion holds for level sets or the Euler characteristic of excursion sets (after changing condition 4).
- Conditions 1.-3. are easy to verify, condition 4. is more challenging but has been proven for some levels (using Morse theory and critical points).

Corollary

Let f be the Bargmann-Fock field.

1. *If* $\ell \in (-\epsilon, 0.64) \cup (1.02, \infty)$ *then*

$$\mathrm{Var}(N_{ES}(\ell, R)) \gtrsim R^2.$$

2. *If* $|\ell| > 1.37$ *then*

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- For fields which are invariant (in distribution) under rotations, we can show that $c'_{ES}(\ell) \neq 0$ when $\ell \in (-\epsilon, \epsilon)$ or $\ell > C$ and $c'_{LS}(\ell) \neq 0$ when $|\ell| > C$.

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Let f be the Random Plane Wave.

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- The same bounds hold for $\ell \neq 0$ such that $c'_{ES}(\ell) \neq 0$ or $c'_{LS}(\ell) \neq 0$.

Definition (Fluctuations)

Let (X_n) be a sequence of random variables and (a_n) a sequence of positive numbers. We say that (X_n) has fluctuations of order at least (a_n) if there exists $c_1, c_2 > 0$ such that for all n large enough and all $u < v$

$$v - u \leq c_1 a_n \ \Rightarrow \ \mathbb{P}(u \leq X_n \leq v) \leq 1 - c_2$$

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Lemma (Chatterjee 2017)

Let (X_n) and (Y_n) be sequences of random variables and (a_n) a sequence of positive numbers. If $X_n - Y_n$ has fluctuations of order at least a_n and $d_{TV}(X_n, Y_n) \to 0$ then X_n has fluctuations of order at least a_n.

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- We wish to apply this lemma to

$$X_R := N_{ES}(\ell, R) \quad \text{and} \quad Y_R := N_{ES}(\ell + \epsilon_R, R) \quad \text{as } R \to \infty.$$

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$$\mathbb{E}(Y_R - X_R) = (c_{ES}(\ell + \epsilon_R) - c_{ES}(\ell))R^2 + o(R^2 \epsilon_R)$$
$$= c'_{ES}(\ell) R^2 \epsilon_R + o(R^2 \epsilon_R).$$

So if $c'_{ES}(\ell) \neq 0$ then $|\mathbb{E}(Y_R - X_R)| \gtrsim R^2 \epsilon_R$.

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- By a Kac-Rice estimate (Muirhead 2019)

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- By the second moment method (Paley-Zygmund inequality) $Y_R - X_R$ has fluctuations of order $R^2\epsilon_R$.

Step 2 (Random Plane Wave)

- The Random Plane Wave has an orthogonal expansion

$$f(x) = \sum_{m \in \mathbb{Z}} a_m J_{|m|}(r) e^{im\theta}$$

where (r, θ) represents x in polar coordinates, the a_m are standard complex Gaussian random variables independent except that $a_m = \overline{a}_{-m}$ and J_m is the m-th Bessel function.

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- Using deterministic bounds on Bessel functions and arguments from Morse theory we show that

$$\mathbb{P}\left(N_{ES}(f,\ell,R) \neq N_{ES}(f_N,\ell,R)\right) \leq c_0 e^{-c_1 R}$$

where $N \approx 2R$ and $f_N(x) = \sum_{|m|\leq N} a_m J_{|m|}(r) e^{im\theta}$. This allows us to work with f or f_N interchangeably (for our purposes).

Step 2 (Random Plane Wave)

- Observe that for $\ell \neq 0$

$$\{f_N = \ell + \epsilon_R\} = \left\{\frac{\ell}{\ell + \epsilon_R} f_N = \ell\right\}$$

and so

$$\begin{aligned} &d_{TV}(N_{ES}(f_N, \ell, R), N_{ES}(f_N, \ell + \epsilon_R, R)) \\ &\quad = d_{TV}\left(N_{ES}(f_N, \ell, R), N_{ES}\left(\frac{\ell}{\ell + \epsilon_R} f_N, \ell, R\right)\right) \\ &\quad \leq d_{TV}\left(f_N, \frac{\ell}{\ell + \epsilon_R} f_N\right) \leq d_{TV}\left((a_m)_{m=-N}^{N}, \frac{\ell}{\ell + \epsilon_R}(a_m)_{m=-N}^{N}\right) \end{aligned}$$

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- Using Pinsker's inequality this is bounded by the Kullback-Liebler divergence between these Gaussian vectors which is explicitly known.
- We then set $\epsilon_R = 1/\sqrt{R}$ so that this distance converges to zero. By Chatterjee's lemma, $N_{ES}(f, \ell, R)$ has fluctuations of order $R^2 \epsilon_R = R^{3/2}$.

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Step 2 (General fields)

- We choose h_R to be a good approximation of 1 on $[0, R]^2$

$$h_R := \frac{1}{4r^2}\mathcal{F}[\mathbb{1}_{[-r,r]^2}] \quad \text{for } r \to 0 \text{ as } R \to \infty$$

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- Using a Morse theory argument, we show that

$$N_{ES}(f, \ell + \epsilon_R, R) = N_{ES}(f - \epsilon_R, \ell, R) = N_{ES}(f - \epsilon_R h_R, \ell, R) + o_{\mathbb{P}}(R^2 \epsilon_R)$$

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- Therefore from Step 1

$$N_{ES}(f - \epsilon_R h_R, \ell, R) - N_{ES}(f, \ell, R)$$

has fluctuations of order $R^2 \epsilon_R$.

Step 2 (General fields)

- By definition of total variation distance

$$d_{TV}(N_{ES}(f,\ell,R), N_{ES}(f-\epsilon_R h_R,\ell,R)) \leq d_{TV}(f, f-\epsilon_R h_R)$$

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$$d_{TV}(f, f-\epsilon_R h_R) \leq c_0 \|\epsilon_R h_R\|_{\mathcal{H}}.$$

where $\mathcal{H}$ is the reproducing kernel Hilbert space generated by κ.

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- Our assumptions on ν allow us to bound $\|\epsilon_R h_R\|_{\mathcal{H}}$. (This bound is smaller when ν is close to a δ mass at the origin).
- For the correct choice of ϵ_R (which depends on ν) this norm tends to 0 so we can apply Chatterjee's lemma to show that $N_{ES}(f,\ell,R)$ has fluctuations of order $R^2\epsilon_R$.

1. Extending lower bounds to all (or almost all) levels and more general spectral measures
2. Considering different variables for Chatterjee's lemma
3. Finding matching upper bounds for variance

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Thank you for listening!