Gaussian Fields and Percolation

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Overview

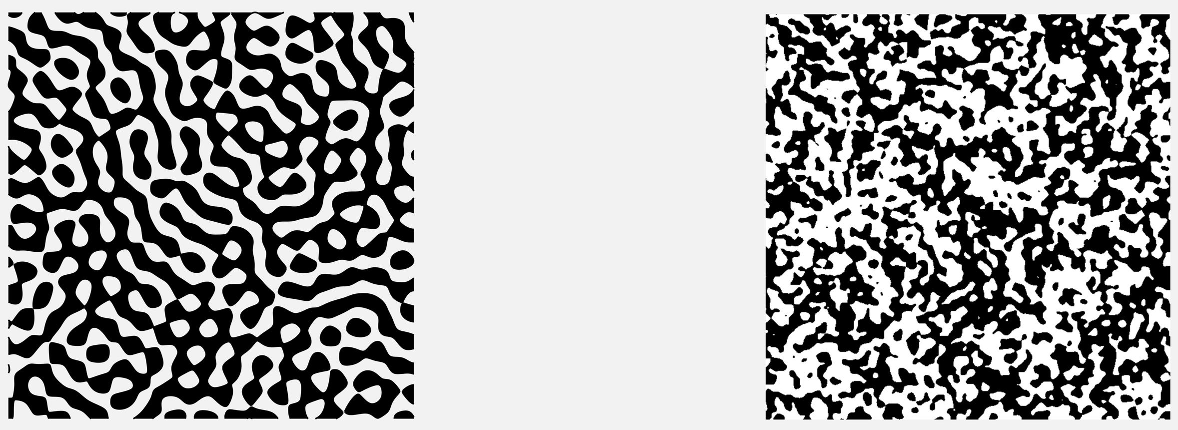
- The geometry of excursion sets of Gaussian fields is important for cosmology, quantum chaos and medical imaging.
- Recently, many classical results from percolation theory (sharp phase transition, noise sensitivity etc) have been proven for excursion sets of general planar Gaussian fields .
- A particular variable of interest is the number of connected components of an excursion set in a large domain. We have derived lower bounds on the variance of this quantity.

Excursion sets of Gaussian fields

A random function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be a (planar) Gaussian field if its distribution at any finite set of points is jointly normal. We assume that f is stationary, that $f \in C^2(\mathbb{R}^2)$ almost surely and that $f(\mathbf{x})$ has mean zero and variance one for each $\mathbf{x} \in \mathbb{R}^2$. We are interested in the geometry of the (upper) excursion sets

$$\left\{\mathbf{x} \in \mathbb{R}^2 \mid f(\mathbf{x}) \geq \ell\right\}$$

for $\ell \in \mathbb{R}$. In particular, we study the number of connected components of these sets contained in a large subset of $\mathbb{R}^2$.



(a) The Random Plane Wave (b) The Bargmann-Fock field

Figure: The zero excursion set for realisations of two different Gaussian fields: f is positive on black regions and negative on white regions. (Credit: Dmitry Beliaev)

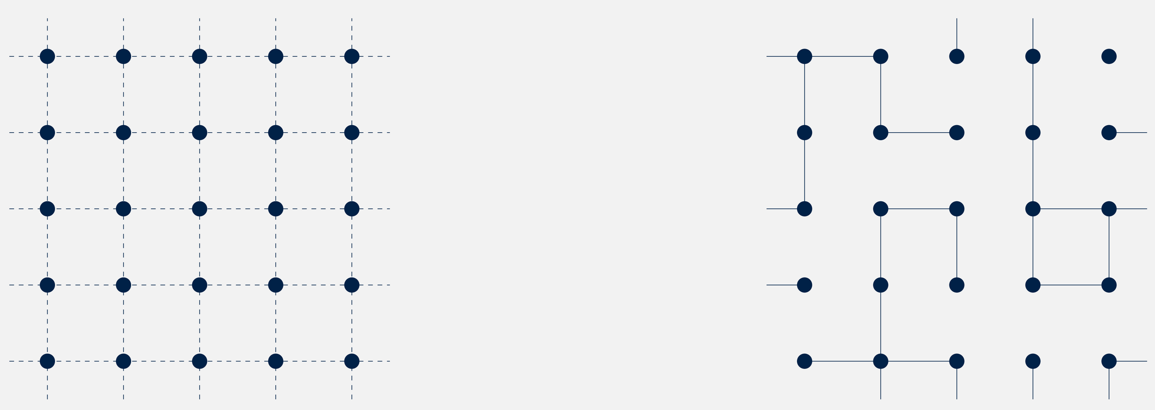
Motivation

Geometric quantities can be used as test statistics for random fields in many different areas and have advantages over more naive approaches such as performing multiple tests on the values of the field.

- Cosmological theories predict that the Cosmic Microwave Background Radiation observed on earth can be well modelled as a realisation of a Gaussian field on the two-dimensional sphere and any deviations from this prediction have potentially important physical implications. This theory can be tested by analysing the number of excursion set components of the field at different levels [7].
- The output of brain imaging scans (PET, FMRI, EEG) can be analysed in a similar way to determine whether stimuli cause spikes in brain activity in certain regions [8]. In this case the test statistic is usually the maximum of the Gaussian field.
- For more details and further applications (in quantum chaos and oceanography), see [1, Chapter 5].

The percolation phase transition

Percolation models are used to understand long range connection properties of random media. Bond percolation on the square lattice is the most classical such model and is defined as follows: form a graph with vertices $\mathbb{Z}^2$ and edges between nearest neighbours, declare each edge to be 'open' independently with probability $p \in [0,1]$ (and closed otherwise), the open sub-graph is a realisation of the model.

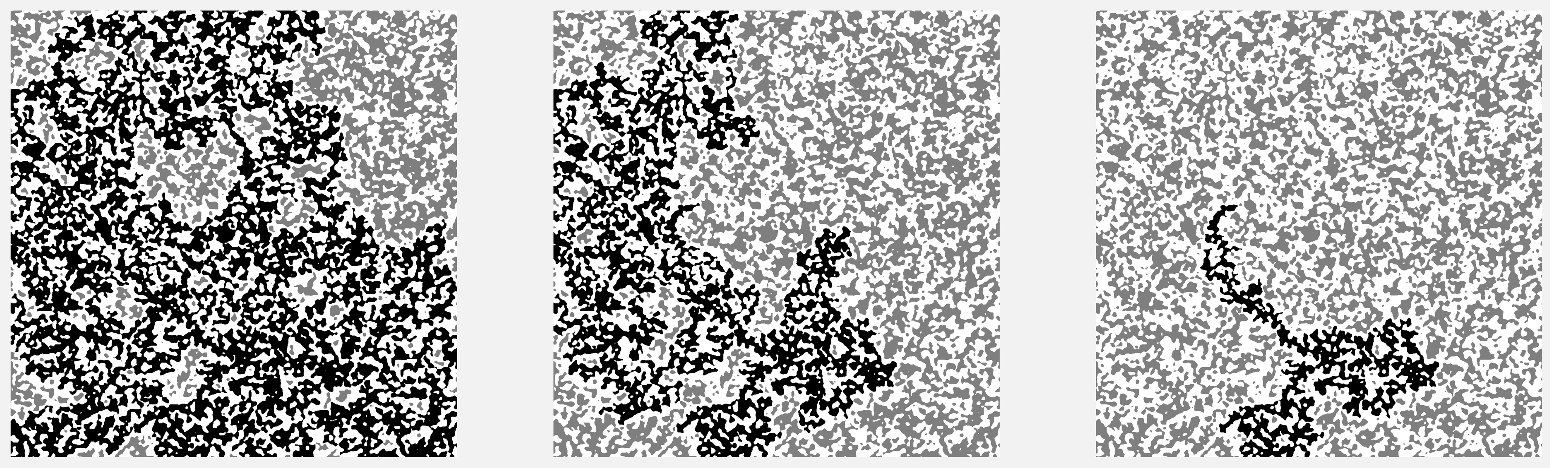


(a) The square lattice (b) A realisation of bond percolation

The most classical result from percolation is a phase transition for long range connections: for $p \leq 1/2$, the model almost surely has only bounded components whereas for $p > 1/2$ it almost surely has a unique unbounded component. Recently, many percolation-type results have been proven for excursion sets of Gaussian fields, including the phase transition.

The percolation phase transition (cont.)

***Theorem* 1** ([5] and references therein)**.**
Let f be a Gaussian field satisfying certain regularity assumptions. For $\ell \geq 0$ the set $\left\{\mathbf{x} \in \mathbb{R}^2 \mid f(\mathbf{x}) \geq \ell\right\}$ almost surely has no unbounded component. For $\ell < 0$, the set $\left\{\mathbf{x} \in \mathbb{R}^2 \mid f(\mathbf{x}) \geq \ell\right\}$ almost surely has a unique unbounded component.



(a) $\ell = -0.1$ (b) $\ell = 0$ (c) $\ell = 0.1$

Figure: The excursion set $\{x \in \mathbb{R}^2 \mid f(x) \geq \ell\}$ for a Gaussian field f at different levels ℓ are in grey. The largest connected component of each set is in black. (Credit: Dmitry Beliaev)

Intuitively this theorem says that at levels below zero, excursion sets have connections on infinite scales, whereas at levels above zero, connections only occur on finite scales. These differences are illustrated in the above figure.

The number of excursion sets

A particular percolation-type variable of interest is the number of connected components of a Gaussian field excursion set in a large domain. The expectation of this quantity is characterised by the following important result.

***Theorem* 2.** *(Nazarov-Sodin [6],[2])*
Let f be an ergodic Gaussian field and let $N(R, \ell)$ be the number of components of $\{x \in \mathbb{R}^2 \mid f(x) \geq \ell\}$ contained in $[-R, R]^2$, then there exists $c_{ES}(\ell) \geq 0$ such that as $R \to \infty$,

$$\frac{N(R,\ell)}{4R^2} \to c_{ES}(\ell) \qquad \textit{almost surely and in } L^1.$$

In particular, the expected number of excursion set components in a domain of area R^2 is of order R^2. The next step in understanding this quantity is to analyse the variance. In work in progress, we show that for generic fields and levels, this variance is of order at least R^2. This is expected to be optimal for most fields, based on analogous results from percolation theory. We also consider a particular Gaussian field known as the Random Plane Wave, which has applications in quantum mechanics [4], and show that in this case the variance is of order at least R^3, suggesting that the order of this variance is non-universal (i.e. it depends on the field).

***Theorem* 3.** *Let f be a Gaussian field satisfying some technical assumptions. If $c'_{ES}(\ell) \neq 0$, then there exists $c > 0$ such that for large R*

$$\mathrm{Var}(N(R,\ell)) \geq cR^2.$$

Let f be the Random Plane Wave, if $\ell \neq 0$ and $c'_{ES}(\ell) \neq 0$ then there exists $c > 0$ such that for large R

$$\mathrm{Var}(N(R,\ell)) \geq cR^3.$$

For more details see [2, 3]

References

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