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Fluctuations of the number of excursion sets of planar Gaussian fields

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Joint work with Dmitry Beliaev and Stephen Muirhead

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University of Oxford

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Conference on random nodal sets

Université de Rennes 1

Oxford Mathematics

Outline

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1. Background and motivation

2. Fluctuation results

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a stationary Gaussian field with mean zero, variance one, covariance function $\kappa : \mathbb{R}^2 \to [-1, 1]$ and spectral measure ν, i.e. for $x, y \in \mathbb{R}^2$

$$\kappa(x) = \mathbb{E}(f(y)f(y + x)) = \int_{\mathbb{R}^2} e^{it\cdot x} d\nu(t).$$

We assume that $\kappa \in C^{6+\epsilon}$ which implies that $f \in C^3(\mathbb{R}^2)$ almost surely.

We are interested in the number of connected components of the (upper) excursion set

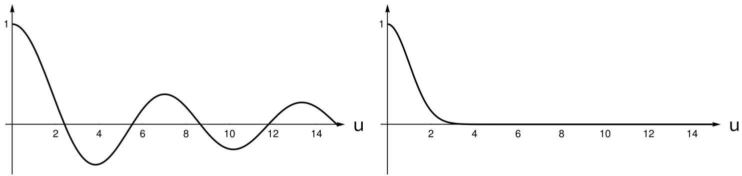
$$\left\{x \in \mathbb{R}^2 \mid f(x) \geq \ell\right\}$$

and level sets

$$\left\{x \in \mathbb{R}^2 \mid f(x) = \ell\right\}$$

in a large domain, for $\ell \in \mathbb{R}$.

1. Random Plane wave
   - $\kappa(x) = J_0(|x|)$ the zero-th Bessel function
   - ν is normalised Lebesgue measure on the unit circle
   - Realisations of f are eigenfunctions of the Laplacian with eigenvalue -1
2. Bargmann-Fock field
   - $\kappa(x) = \exp(-|x|^2/2)$
   - $\nu(t) = \exp(-|t|^2/2)\,dt$
   - scaling limit of random homogeneous polynomials on $\mathbb{R}P^2$



(a) $u \mapsto J_0(u)$ (b) $u \mapsto \exp(-u^2/2)$

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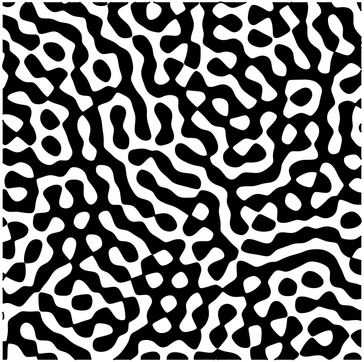
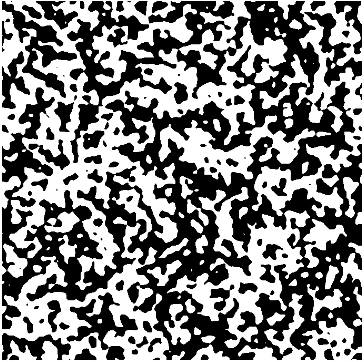


(a) Nodal set (i.e. zero level set) of Random Plane Wave



(b) Nodal set of Bargmann-Fock field

Figure: f is positive on black regions and negative on white regions.

Let $N_{ES}(\ell, R)$ be the number of components of $\{f \geq \ell\}$ in $[0, R]^2$ and $N_{LS}(\ell, R)$ the corresponding number for $\{f = \ell\}$.

Theorem (Nazarov-Sodin 2016, Kurlberg-Wigman 2017)

There exists $c_{ES}(\ell) \geq 0$ such that

$$\mathbb{E}(N_{ES}(\ell, R)) = c_{ES}(\ell)R^2 + O(R) \qquad \text{as } R \to \infty.$$

If f is ergodic then

$$N_{ES}(\ell, R)/R^2 \to c_{ES}(\ell)$$

a.s. and in L^1 as $R \to \infty$.
The same result holds if N_{ES} and c_{ES} are replaced by N_{LS} and c_{LS} respectively.

Number of excursion/level sets

First moment results

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1. The proof relies on an ergodic argument and the fact that the number of components is 'semi-local'.
2. Further properties of the functional c_{LS} have been studied (differentiability, lower/upper bounds, monotonicity).

- Trivial result

$$\mathrm{Var}(N_{ES}(\ell, R)) \leq CR^4$$

Variance of the number of excursion sets

Previous results

- Trivial result

$$\mathrm{Var}(N_{ES}(\ell, R)) \leq CR^4$$

- (Nazarov-Sodin 2009) For random spherical harmonics ('Random Plane Wave on the sphere')

$$\mathrm{Var}(N_{LS}(0, f_n)) \leq Cn^{4-2/15}$$

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- (Nazarov-Sodin, announced) For random spherical harmonics

$$cn^{\sigma} \leq \mathrm{Var}(N_{LS}(0, f_n))$$

  for some $\sigma > 0$.

Outline

Theorem 1

Fix $\ell \in \mathbb{R}$ and let f be a Gaussian field such that:

1. $|\partial^\alpha \kappa(x)| \leq c|x|^{-(1+\epsilon)}$ *for* $|\alpha| \leq 3$ *and some* $c, \epsilon > 0$
2. ν *has a density* ρ *on a neighbourhood of* 0 *which is bounded away from* 0
3. *There exists* $c_1, c_2 > 0$ *s.t.* $\mathbb{P}(f \in Arm_0(r, R)) \leq c_1(r/R)^{c_2}$
4. $c'_{ES}(\ell) \neq 0$

then for some $c > 0$

$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^2.$$

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- Sufficient conditions for Assumption 3 are given in (Muirhead-Vanneuville 18) and (Rivera 19).
- Sufficient conditions for Assumption 4 (at some levels) are given in (Beliaev-M.-Muirhead 19)

Corollary

Let f be the Bargmann-Fock field.

1. *If* $\ell \in (-\epsilon, 0.64) \cup (1.02, \infty)$ *then*

$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^2.$$

2. *If* $\ell \in (-\infty, -1.37) \cup (1.37, \infty)$ *then*

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- For fields which are isotropic, we can show that $c_{ES}'(\ell) \neq 0$ when $\ell \in (-\epsilon, \epsilon)$ or $\ell > C$ and $c_{LS}'(\ell) \neq 0$ when $|\ell| > C$.

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Theorem 2

Fix $\ell \in \mathbb{R}$ and let f be a Gaussian field such that:

1. $|\partial^\alpha \kappa(x)| \to 0$ *as* $|x| \to \infty$ *for* $|\alpha| \leq 2$
2. ν *has a density* ρ *on a neighbourhood of* 0 *which is bounded away from* 0
3. $g(r) := \inf_{x \in B(2r)} \rho(x) \to \infty$ *as* $r \to 0$
4. $c'_{ES}(\ell) \neq 0$

then for some $c > 0$

$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^2 g(1/R).$$

Theorem 2

Fix $\ell \in \mathbb{R}$ and let f be a Gaussian field such that:

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- The bound $R^2 g(1/R)$ interpolates between R^2 and $o(R^4)$ since ρ is integrable.

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- The same conclusion holds for level sets (replacing N_{ES} and c_{ES} with N_{LS} and c_{LS}).

Let D_+ denote the right lower Dini-derivative:

$$D_+h(x) = \liminf_{\epsilon \downarrow 0} (h(x+\epsilon) - h(x))/\epsilon.$$

Theorem 3

Let f be the Random Plane Wave and $\ell \neq 0$, if $D_+ c_{ES}(\ell) > 0$ then there exists $c > 0$ such that

$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^3.$$

The same result holds if N_{ES} and c_{ES} are replaced by N_{LS} and c_{LS}.

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Corollary

Let f be the Random Plane Wave.

1. *If* $\ell \in (-\infty, 0) \cup (0, 0.87) \cup [1, \infty)$ *then there exists* $c > 0$ *such that*
$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^3.$$
2. *If* $\ell \in (-\infty, -1] \cup [1, \infty)$ *then there exists* $c > 0$ *such that*
$$\mathrm{Var}(N_{LS}(\ell, R)) \geq cR^3.$$

Definition (Fluctuations)

Let (X_n) be a sequence of random variables and (a_n) a sequence of positive numbers. We say that (X_n) has fluctuations of order at least (a_n) if there exists $c_1, c_2 > 0$ such that for all n large enough and all $u < v$

$$v - u \leq c_1 a_n \Rightarrow \mathbb{P}(u \leq X_n \leq v) \leq 1 - c_2$$

If (X_n) has fluctuations of order (a_n) then $\mathrm{Var}(X_n) \geq c a_n^2$.

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Lemma (Chatterjee 2017)

Let (X_n) and (Y_n) be sequences of random variables and (a_n) a sequence of positive numbers. If $X_n - Y_n$ has fluctuations of order at least a_n and $d_{TV}(X_n, Y_n) \to 0$ then X_n has fluctuations of order at least a_n.

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- We wish to apply this lemma to

$$X_R := N_{ES}(\ell, R) \quad \text{and} \quad Y_R := N_{ES}(\ell + \epsilon_R, R) \quad \text{as } R \to \infty.$$

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- By extending Nazarov and Sodin's result

$$\begin{aligned}\mathbb{E}(Y_R - X_R) &= (c_{ES}(\ell + \epsilon_R) - c_{ES}(\ell))R^2 + o(R^2\epsilon_R) \\ &= c'_{ES}(\ell)R^2\epsilon_R + o(R^2\epsilon_R).\end{aligned}$$

So if $c'_{ES}(\ell) \neq 0$ then $|\mathbb{E}(Y_R - X_R)| \gtrsim R^2\epsilon_R$.

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  So if $c'_{ES}(\ell) \neq 0$ then $|\mathbb{E}(Y_R - X_R)| \gtrsim R^2 \epsilon_R$.

- By a Kac-Rice estimate (Muirhead 2019)

$$\begin{aligned}\mathbb{E}((Y_R - X_R)^2) &\leq \mathbb{E}(\#\{\text{Critical points in } [0,R]^2 \text{ at level } [\ell, \ell + \epsilon_R]\}^2) \\ &\leq c_0(R^4 \epsilon_R^2 + R^2 \epsilon_R).\end{aligned}$$

Variance of the number of excursion sets

Lower bound: proof

Step 1

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- By extending Nazarov and Sodin's result

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- By a Kac-Rice estimate (Muirhead 2019)

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- By the second moment method (Paley-Zygmund inequality) $Y_R - X_R$ has fluctuations of order $R^2\epsilon_R$.

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Step 2 (General fields)

- We choose h_R to be a good approximation of 1 on $[0, R]^2$

$$h_R := \frac{1}{4r^2}\mathcal{F}[\mathbb{1}_{[-r,r]^2}] \quad \text{for } r \to 0 \text{ as } R \to \infty$$

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- Using a Morse theory argument, we show that

$$N_{ES}(f, \ell + \epsilon_R, R) = N_{ES}(f - \epsilon_R, \ell, R) = N_{ES}(f - \epsilon_R h_R, \ell, R) + o_{\mathbb{P}}(R^2 \epsilon_R)$$

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- Therefore from Step 1

$$N_{ES}(f-\epsilon_R h_R,\ell,R) - N_{ES}(f,\ell,R)$$

has fluctuations of order $R^2\epsilon_R$.

Step 2 (General fields)

- By definition of total variation distance

$$d_{TV}(N_{ES}(f,\ell,R), N_{ES}(f-\epsilon_R h_R,\ell,R)) \leq d_{TV}(f, f-\epsilon_R h_R)$$

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- A Cameron-Martin argument (Muirhead-Vanneuville 2019) shows that

$$d_{TV}(f, f-\epsilon_R h_R) \leq c_0 \|\epsilon_R h_R\|_{\mathcal{H}}.$$

where $\mathcal{H}$ is the reproducing kernel Hilbert space generated by κ.

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- Our assumptions on ν allow us to bound $\|\epsilon_R h_R\|_{\mathcal{H}}$. (This bound is smaller when ν is close to a δ mass at the origin).
- For the correct choice of ϵ_R (which depends on ν) this norm tends to 0 so we can apply Chatterjee's lemma to show that $N_{ES}(f,\ell,R)$ has fluctuations of order $R^2\epsilon_R$.

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Step 2 (Random Plane Wave)

- The Random Plane Wave has an orthogonal expansion

$$f(x) = \sum_{m \in \mathbb{Z}} a_m J_{|m|}(r) e^{im\theta}$$

where (r, θ) represents x in polar coordinates, the a_m are standard complex Gaussian random variables independent except that $a_m = \overline{a}_{-m}$ and J_m is the m-th Bessel function.

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- Using deterministic bounds on Bessel functions and arguments from Morse theory we show that

$$\mathbb{E}\left(N_{ES}(f,\ell,R) \neq N_{ES}(f_N,\ell,R)\right) \leq C$$

where $N \approx 2R$ and $f_N(x) = \sum_{|m|\leq N} a_m J_{|m|}(r) e^{im\theta}$. This allows us to work with f or f_N interchangeably (for our purposes).

Step 2 (Random Plane Wave)

- Observe that for $\ell \neq 0$

$$\{f_N \geq \ell + \epsilon_R\} = \left\{ \frac{\ell}{\ell + \epsilon_R} f_N \geq \ell \right\}$$

and so

$$\begin{aligned} & d_{TV}(N_{ES}(f_N, \ell, R), N_{ES}(f_N, \ell + \epsilon_R, R)) \\ & = d_{TV}\left(N_{ES}(f_N, \ell, R), N_{ES}\left(\frac{\ell}{\ell + \epsilon_R} f_N, \ell, R \right) \right) \\ & \leq d_{TV}\left(f_N, \frac{\ell}{\ell + \epsilon_R} f_N \right) \leq d_{TV}\left((a_m)_{m=-N}^{N}, \frac{\ell}{\ell + \epsilon_R} (a_m)_{m=-N}^{N} \right) \end{aligned}$$

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$$= d_{TV}\left(N_{ES}(f_N, \ell, R), N_{ES}\left(\frac{\ell}{\ell + \epsilon_R} f_N, \ell, R \right) \right)$$
$$\leq d_{TV}\left(f_N, \frac{\ell}{\ell + \epsilon_R} f_N \right) \leq d_{TV}\left((a_m)_{m=-N}^{N}, \frac{\ell}{\ell + \epsilon_R} (a_m)_{m=-N}^{N} \right)$$
- Using Pinsker's inequality this is bounded by the Kullback-Liebler divergence between these Gaussian vectors which is explicitly known.
- We then set $\epsilon_R = 1/\sqrt{R}$ so that this distance converges to zero. By Chatterjee's lemma, $N_{ES}(f, \ell, R)$ has fluctuations of order $R^2 \epsilon_R = R^{3/2}$.

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1. Our proof relies on comparing the number of excursion/level sets at different levels. This is heuristically similar to results for local functionals which are 'driven by L^2-norm fluctuations'.
2. We expect these lower bounds to be of the correct order for general levels (based on the percolation analogy and local geometric functionals of RPW).
3. Our proof necessarily fails at critical levels of c_{ES} and c_{LS} (including for nodal sets). How many such levels are there? Does the conclusion also fail at these levels? If so, what is the order of variance at these levels?

Conjecture

Let f satisfy the conditions of Theorem 1 with ρ bounded above near 0. For $\ell \in \mathbb{R}$ there exists $c_{fluc}(\ell) > 0$ such that

$$\mathrm{Var}(N_{ES}(R,\ell)) \sim c_{fluc}(\ell) R^2,$$

and the same conclusion is true for $N_{LS}(R,\ell)$.

Conjecture

Let f be the Random Plane Wave, there exists a (possibly finite) set $\mathcal{L} \subset \mathbb{R}$ such that the following holds. For $\ell \in \mathbb{R} \backslash \mathcal{L}$ there exists $c_{fluc}(\ell) > 0$ such that

$$\mathrm{Var}(N_{ES}(R,\ell)) \sim c_{fluc}(\ell) R^3.$$

For $\ell \in \mathcal{L}$

$$\mathrm{Var}(N_{ES}(R,\ell)) = o(R^3) \quad \text{as } R \to \infty.$$

The same conclusion is true for $N_{LS}(R,\ell)$ (with a different set $\mathcal{L}$).

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1. Proving $c'_{ES}(\ell) \neq 0$ or $c'_{LS}(\ell) \neq 0$ for a wider range of levels,
2. Extending the proof beyond the case of positive spectral density around 0,
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Thank you for listening!