Smooth Gaussian fields and percolation

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Outline

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Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^3, stationary Gaussian field with mean zero, variance one, covariance function $\kappa : \mathbb{R}^2 \to [-1, 1]$ and spectral measure ν, i.e. for $x, y \in \mathbb{R}^2$

$$\kappa(x) = \mathbb{E}(f(y)f(y + x)) = \int_{\mathbb{R}^2} e^{it\cdot x} d\nu(t).$$

We are interested in the excursion sets

$$\left\{x \in \mathbb{R}^2 \mid f(x) \geq \ell\right\}$$

and level sets

$$\left\{x \in \mathbb{R}^2 \mid f(x) = \ell\right\}$$

in a large domain, for $\ell \in \mathbb{R}$.

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1) Cosmology

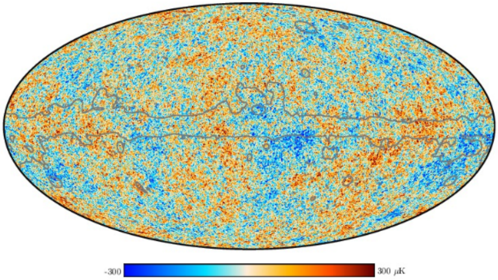

Figure: Fluctuations of the Cosmic Microwave Background Radiation (CMBR) (Source: Planck 2018).

Theory predicts that the CMBR can be modelled as a Gaussian field on the sphere. This prediction can be tested statistically using geometric properties of excursion sets.

2) Quantum Chaos

Conjecture (Berry 1977)

Let Ω be a Riemannian 2-manifold with 'chaotic' dynamics. Let ϕ_λ be an eigenfunction of the Laplacian with eigenvalue λ, then as $\lambda \to \infty$, ϕ_λ is well modelled by 'Gaussian monochromatic random waves'.

The local scaling limit of 'Gaussian monochromatic random waves' on a manifold is a Gaussian field on $\mathbb{R}^2$ known as the Random Plane Wave.

3) Hilbert's 16-th problem

One can study the components of 'typical' real algebraic hypersurfaces by placing a canonical Gaussian measure on homogeneous polynomials in projective space. As the degree of the polynomials increase, the scaling limit of these measures is known as the Bargmann-Fock field.

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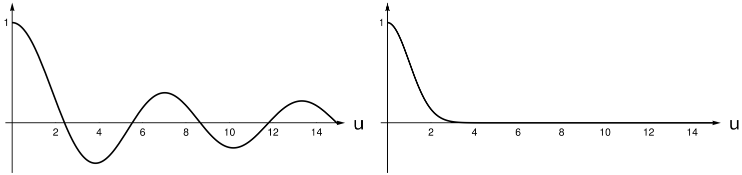
1. Random Plane wave
   - $\kappa(x) = J_0(|x|)$ the 0-th Bessel function
   - ν is normalised Lebesgue measure on the unit circle
   - Realisations of f are eigenfunctions of the Laplacian with eigenvalue -1
2. Bargmann-Fock field
   - $\kappa(x) = \exp(-|x|^2/2)$
   - $\nu(t) = \exp(-|t|^2/2)\, dt$
   - scaling limit of random homogeneous polynomials on $\mathbb{R}P^2$



(a) $u \mapsto J_0(u)$ (b) $u \mapsto \exp(-u^2/2)$

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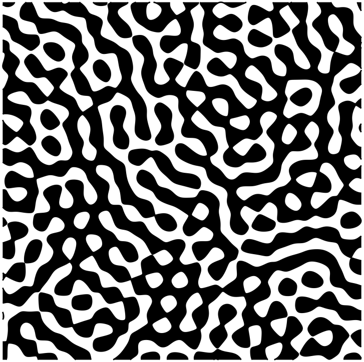
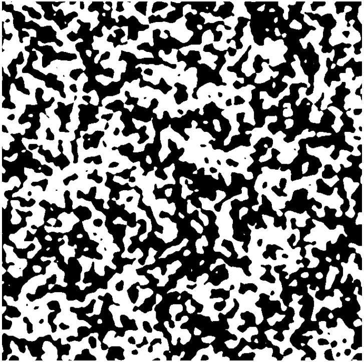


(a) Nodal set (i.e. zero level set) of Random Plane Wave



(b) Nodal set of Bargmann-Fock field

Figure: f is positive on black regions and negative on white regions.

Percolation models
The standard example

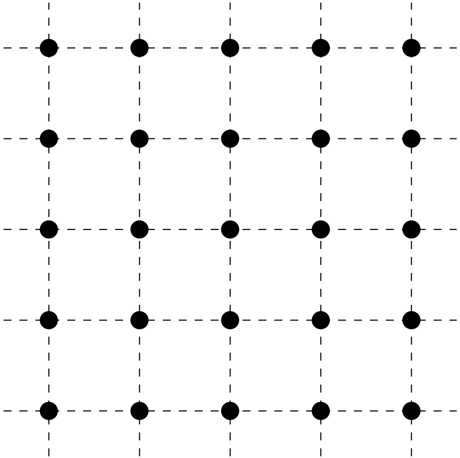


Figure: Bond percolation on the square lattice

Percolation theory

The standard model

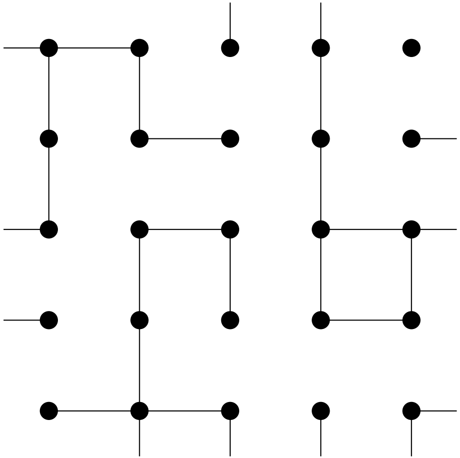


Figure: Bond percolation on the square lattice

Theorem

For Bernoulli percolation on $\mathbb{Z}^2$ *with parameter p, if* $p \leq 1/2$ *then a.s. there is no infinite open connected component (Harris 1960). If* $p > 1/2$ *then a.s. there exists a unique infinite open connected component (Kesten 1980).*

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Let $C_{[a,b]\times[c,d]}$ be the event that there exists an open path in $[a, b] \times [c, d]$ joining the left and right sides of the rectangle.

Theorem

If $p = 1/2$ then for each $c > 0$ there exists $c_1 > 0$ such that

$$c_1 < \mathbb{P}(C_{[0,R]\times[0,cR]}) < 1 - c_1 \qquad \text{(RSW)}$$

for all $R > 0$. If $p > 1/2$ then for each $c > 0$ there exists $c_2 > 0$ such that

$$\mathbb{P}(C_{[0,R]\times[0,cR]}) > 1 - e^{-c_2 R} \qquad \text{(Kesten 1980)}$$

Heuristic connection between Gaussian fields and percolation

The Bogomolny-Schmit conjecture

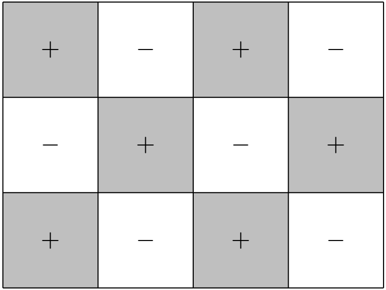

Figure: Approximating nodal lines of the Random Plane Wave by a square grid.

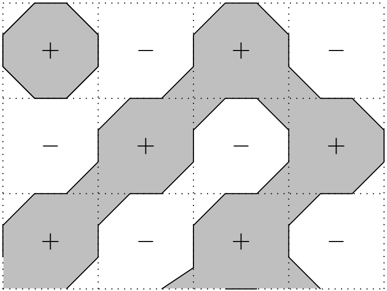

Figure: Adjusting the corners of the grid.

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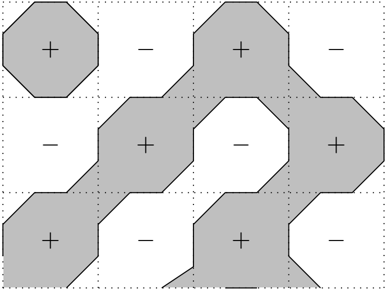

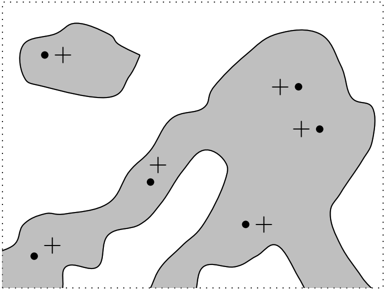

Figure: The corresponding Gaussian excursion set.

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Conjecture (Bogomolny-Schmit 2001)

The nodal set of the Random Plane Wave (i.e. the set $\{f = 0\}$*) can be modelled by critical Bernoulli percolation on the square lattice.*
In particular, for $R > 0$ *sufficiently large*

$$N(R) \approx \mathcal{N}\left(\mu R^2, \sigma^2 R^2\right)$$

where $N(R)$ *is the number of components of* $\{f = 0\}$ *in* $[0, R]^2$ *and* μ, σ^2 *are explicitly known constants.*
They later extend the conjecture to non-zero levels without deriving the analogous mean and variance.

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- Numerical results indicate that the prediction for μ is inaccurate (by about 5%).
- However the probability of ‘crossing events’ for the Random Plane Wave match those for percolation extremely well numerically.

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Gaussian fields

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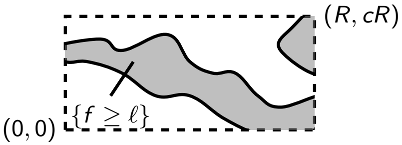
- (Beliaev-Muirhead 2018) As above for $\beta > 16$
- (Rivera-Vanneuville 2018) As above for $\beta > 4$.

Let $C_\ell(R,\alpha)$ be the event that there exists a left-right crossing in $\{f \geq \ell\}$ of $[0,R] \times [0,\alpha R]$.



- (Rivera-Vanneuville 2018) For the Bargmann-Fock field, for each $\alpha > 0$, $\ell < 0$ there exists $c > 0$ such that

$$\mathbb{P}\left(C_\ell(R,\alpha)\right) > 1 - e^{-cR}$$

Hence for $\ell < 0$, $\{f \geq \ell\}$ almost surely has a unique unbounded component.

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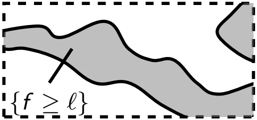

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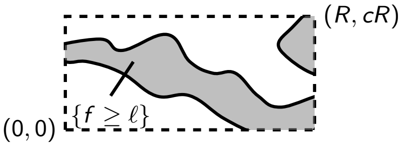

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- (Rivera 2019) Removes the technical conditions above.

- There has been impressive progress on this topic recently!

Gaussian fields
Takeaways from literature

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- These results depend heavily on percolation methods, so positive correlations and fast decay of correlations are essential. Hence these results fail for some interesting fields including the Random Plane Wave! (Although they are believed to be true.)

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- These results depend heavily on percolation methods, so positive correlations and fast decay of correlations are essential. Hence these results fail for some interesting fields including the Random Plane Wave! (Although they are believed to be true.)
- It would be nice to have more of an *explanation* for similarities between Guassian fields and discrete percolation models.
- Convergence of level lines to Schramm-Loewner Evolution could provide such an explanation (this has been tested numerically, and seems plausible)

Outline

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1. Overview of Gaussian fields and percolation

2. Number of excursion/level sets

As before, let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^3-smooth, stationary Gaussian field. Let $N_{ES}(\ell, R)$ be the number of components of $\{f \geq \ell\}$ in $[0, R]^2$ and $N_{LS}(\ell, R)$ the corresponding number for $\{f = \ell\}$.

What are the statistics of N_{ES} and N_{LS} as $R \to \infty$?

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What are the statistics of N_{ES} and N_{LS} as $R \to \infty$?

We can compare this with percolation:

Theorem (Cox-Grimmett 1984, Zhang 2000)

Let K_n be the number of open components of Bernoulli bond percolation on $\mathbb{Z}^d$ in $[0, n]^d$. For $0 < p < 1$ there exists $\kappa(p), \sigma(p) > 0$ such that

$$\frac{K_n - \kappa(p) n^d}{\sigma(p) n^{d/2}} \xrightarrow{d} \mathcal{N}(0, 1)$$

In particular $\mathbb{E}(K_n) \sim n^d$ and $Var(K_n) \sim n^d$.

Let $N_{ES}(\ell, R)$ be the number of components of $\{f \geq \ell\}$ in $[0, R]^2$ and $N_{LS}(\ell, R)$ the corresponding number for $\{f = \ell\}$.

Theorem (Nazarov-Sodin 2016, Kurlberg-Wigman 2017)

There exists $c_{ES}(\ell) \geq 0$ such that

$$\mathbb{E}(N_{ES}(\ell, R)) = c_{ES}(\ell)R^2 + O(R) \qquad \text{as } R \to \infty.$$

If f is ergodic then

$$N_{ES}(\ell, R)/R^2 \to c_{ES}(\ell)$$

a.s. and in L^1 as $R \to \infty$.
The same result holds if N_{ES} and c_{ES} are replaced by N_{LS} and c_{LS} respectively.

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1. The proof relies on an ergodic argument and the fact that the number of components is 'semi-local'.
2. Further properties of the functional c_{ES} have been studied (differentiability, lower/upper bounds, monotonicity).

- Based on the percolation analogy, we might expect

$$\mathrm{Var}(N_{ES}(\ell, R)) \sim R^2$$

Number of excursion/level sets

Variance results

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- (Nazarov-Sodin, announced) For random spherical harmonics (and other spherical fields)

$$\mathrm{Var}(N_{LS}(0, f_n)) \geq cn^{\sigma}$$

  for some $\sigma > 0$. (More importantly, they partially justify the Bogomolny-Schmit heuristics)

Theorem 1

Fix $\ell \in \mathbb{R}$ and let f be a (stationary, $C^{3+\epsilon}$) Gaussian field such that:

1. $|\partial^\alpha \kappa(x)| \leq c|x|^{-(1+\epsilon)}$ *for* $|\alpha| \leq 3$ *and some* $c, \epsilon > 0$
2. ν *has a density on a neighbourhood of* 0 *which is bounded away from* 0
3. *There exists* $c_1, c_2 > 0$ *s.t.* $\mathbb{P}(f \in Arm_0(r, R)) \leq c_1(r/R)^{c_2}$
4. $c'_{ES}(\ell) \neq 0$

then for some $c > 0$

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- Sufficient conditions for Assumption 3 are given in (Muirhead-Vanneuville 18) and (Rivera 19).
- Sufficient conditions for Assumption 4 (at some levels) are given in (Beliaev-M.-Muirhead 19)

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Corollary

Let f be the Bargmann-Fock field.

1. *If* $\ell \in (-\epsilon, 0.64) \cup (1.02, \infty)$ *then*
$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^2.$$
2. *If* $|\ell| > 1.37$ *then*
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- For fields which are isotropic, we can show that $c'_{ES}(\ell) \neq 0$ when $\ell \in (-\epsilon, \epsilon)$ or $\ell > C$ and $c'_{LS}(\ell) \neq 0$ when $|\ell| > C$.

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Theorem 2

Fix $\ell \in \mathbb{R}$ and let f be a Gaussian field such that:

1. $|\partial^\alpha \kappa(x)| \to 0$ *as* $|x| \to \infty$ *for* $|\alpha| \leq 2$
2. ν *has a density* ρ *on a neighbourhood of* 0 *which is bounded away from* 0
3. $g(r) := \inf_{x \in B(2r)} \rho(x) \to \infty$ *as* $r \to 0$
4. $c'_{ES}(\ell) \neq 0$

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$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^2 g(1/R).$$

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- The bound $R^2 g(1/R)$ interpolates between R^2 and $o(R^4)$ since ρ is integrable.

Theorem 2

Fix $\ell \in \mathbb{R}$ and let f be a Gaussian field such that:

1. $|\partial^\alpha \kappa(x)| \to 0$ *as* $|x| \to \infty$ *for* $|\alpha| \leq 2$
2. ν *has a density* ρ *on a neighbourhood of* 0 *which is bounded away from* 0
3. $g(r) := \inf_{x \in B(2r)} \rho(x) \to \infty$ *as* $r \to 0$
4. $c'_{ES}(\ell) \neq 0$

then for some $c > 0$

$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^2 g(1/R).$$

- The bound $R^2 g(1/R)$ interpolates between R^2 and $o(R^4)$ since ρ is integrable.
- The same conclusion holds for level sets (replacing N_{ES} and c_{ES} with N_{LS} and c_{LS}).

Let D_+ denote the right lower Dini-derivative:

$$D_+ h(x) = \liminf_{\epsilon \downarrow 0} (h(x+\epsilon) - h(x))/\epsilon.$$

Theorem 3

Let f be the Random Plane Wave and $\ell \neq 0$, if $D_+ c_{ES}(\ell) > 0$ then there exists $c > 0$ such that

$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^3.$$

The same result holds if N_{ES} and c_{ES} are replaced by N_{LS} and c_{LS}.

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Corollary

Let f be the Random Plane Wave.

1. *If* $\ell \in (-\infty, 0) \cup (0, 0.87) \cup [1, \infty)$ *then there exists* $c > 0$ *such that*

$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^3.$$

2. *If* $\ell \in (-\infty, -1] \cup [1, \infty)$ *then there exists* $c > 0$ *such that*

$$\mathrm{Var}(N_{LS}(\ell, R)) \geq cR^3.$$

Comments

- We think these bounds should be sharp in most cases!
- Theorem 1 reiterates the similarity of Bernoulli percolation and 'fast decay' fields
- Theorem 2 contradicts some parts of the Bogomolny-Schmit conjecture (but not the most important case: the zero level)

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- We think these bounds should be sharp in most cases!
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Questions

- How general is the condition $c_{ES}'(\ell) \neq 0$? What happens at levels for which $c_{ES}'(\ell) = 0$?
- Can the conditions of these theorems be generalised?

Definition (Fluctuations)

Let (X_n) be a sequence of random variables and (a_n) a sequence of positive numbers. We say that (X_n) has fluctuations of order at least (a_n) if there exists $c_1, c_2 > 0$ such that for all n large enough and all $u < v$

$$v - u \leq c_1 a_n \Rightarrow \mathbb{P}(u \leq X_n \leq v) \leq 1 - c_2$$

If (X_n) has fluctuations of order (a_n) then $\mathrm{Var}(X_n) \geq c a_n^2$.

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Lemma (Chatterjee 2017)

Let (X_n) and (Y_n) be sequences of random variables and (a_n) a sequence of positive numbers. If $X_n - Y_n$ has fluctuations of order at least a_n and $d_{TV}(X_n, Y_n) \to 0$ then X_n has fluctuations of order at least a_n.

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- We wish to apply this lemma to

$$X_R := N_{ES}(\ell, R) \quad \text{and} \quad Y_R := N_{ES}(\ell + \epsilon_R, R) \quad \text{with } \epsilon_R \to 0 \text{ as } R \to \infty.$$

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Step 1

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- By extending Nazarov and Sodin's result

$$\mathbb{E}(Y_R - X_R) = (c_{ES}(\ell + \epsilon_R) - c_{ES}(\ell))R^2 + o(R^2 \epsilon_R)$$
$$= c'_{ES}(\ell)R^2 \epsilon_R + o(R^2 \epsilon_R).$$

So if $c'_{ES}(\ell) \neq 0$ then $|\mathbb{E}(Y_R - X_R)| \gtrsim R^2 \epsilon_R$.

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- By a Kac-Rice estimate (Muirhead 2019)

$$\mathbb{E}((Y_R - X_R)^2) \leq \mathbb{E}(\#\{\text{Critical points in } [0,R]^2 \text{ at level } [\ell, \ell + \epsilon_R]\}^2)$$
$$\leq c_0 (R^4 \epsilon_R^2 + R^2 \epsilon_R).$$

Variance of the number of excursion sets
Lower bound: proof

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- By extending Nazarov and Sodin's result

$$\begin{aligned}\mathbb{E}(Y_R - X_R) &= (c_{ES}(\ell + \epsilon_R) - c_{ES}(\ell))R^2 + o(R^2\epsilon_R)\\ &= c'_{ES}(\ell)R^2\epsilon_R + o(R^2\epsilon_R).\end{aligned}$$

  So if $c'_{ES}(\ell) \neq 0$ then $|\mathbb{E}(Y_R - X_R)| \gtrsim R^2\epsilon_R$.
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- By the second moment method (Paley-Zygmund inequality) $Y_R - X_R$ has fluctuations of order $R^2\epsilon_R$.

Step 2 (General fields)

- By definition of total variation distance

$$d_{TV}(N_{ES}(f,\ell,R), N_{ES}(f-\epsilon_R,\ell,R)) \leq d_{TV}(f, f-\epsilon_R)$$

(where we restrict all functions to $[0,R]^2$).

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  (where we restrict all functions to $[0,R]^2$).
- Let $h_R \approx \mathbb{1}_{[0,R]^2}$, then a Cameron-Martin argument (Muirhead-Vanneuville 2019) shows that

$$d_{TV}(f, f-\epsilon_R h_R) \leq c_0 \|\epsilon_R h_R\|_{\mathcal{H}}.$$

  where $\mathcal{H}$ is the reproducing kernel Hilbert space generated by κ. Our assumptions on ν allow us to bound $\|\epsilon_R h_R\|_{\mathcal{H}}$. For the correct choice of ϵ_R this norm tends to 0.

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Step 2 (Random Plane Wave)

- The Random Plane Wave has an orthogonal expansion in polar coordinates

$$\text{For } x = (r, \theta) \in [0, N/2]^2 \qquad f(x) \approx f_N(x) := \sum_{|m| \leq N} a_m J_{|m|}(r) e^{im\theta}$$

where the a_m are complex Gaussian random variables and J_m is the m-th Bessel function.

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where the a_m are complex Gaussian random variables and J_m is the m-th Bessel function.

- Then for $\ell \neq 0$ and $N \approx R$

$$\begin{aligned} & d_{TV}(N_{ES}(f_N, \ell, R), N_{ES}(f_N, \ell + \epsilon_R, R)) \\ & \quad = d_{TV}\left(N_{ES}(f_N, \ell, R), N_{ES}\left(\frac{\ell}{\ell + \epsilon_R} f_N, \ell, R\right)\right) \\ & \quad \leq d_{TV}\left(f_N, \frac{\ell}{\ell + \epsilon_R} f_N\right) \leq d_{TV}\left((a_m)_{m=-N}^{N}, \frac{\ell}{\ell + \epsilon_R} (a_m)_{m=-N}^{N}\right) \end{aligned}$$

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- This can be bounded explicitly using Pinsker's inequality.

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1. Our lower bounds are conceptually due to 'shifts in the level of the field'. This is heuristically similar to results for some other geometric functionals (e.g. length of level sets) which are 'driven by L^2-norm fluctuations'.
2. If these 'level shifts' are the dominant source of fluctuations, we may expect lower variance whenever $c_{ES}'(\ell) = 0$.
3. We expect there are only a small number of levels satisfying these conditions (perhaps 1-3), but have very limited justification for this.

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Conjecture

Let f satisfy the conditions of Theorem 1 with ρ bounded above near 0. For $\ell \in \mathbb{R}$ there exists $c_{fluc}(\ell) > 0$ such that

$$\mathrm{Var}(N_{ES}(R,\ell)) \sim c_{fluc}(\ell) R^2,$$

and the same conclusion is true for $N_{LS}(R,\ell)$.

Conjecture

Let f be the Random Plane Wave, there exists a (possibly finite) set $\mathcal{L} \subset \mathbb{R}$ such that the following holds. For $\ell \in \mathbb{R} \backslash \mathcal{L}$ there exists $c_{fluc}(\ell) > 0$ such that

$$\mathrm{Var}(N_{ES}(R,\ell)) \sim c_{fluc}(\ell) R^3.$$

For $\ell \in \mathcal{L}$

$$\mathrm{Var}(N_{ES}(R,\ell)) = o(R^3) \quad \text{as } R \to \infty.$$

The same conclusion is true for $N_{LS}(R,\ell)$ (with a different set $\mathcal{L}$).

1. Theorem 3 only depends on the fact that Random Plane Wave is specified by N parameters in a domain of area N^2, so this could be extended to other fields with a similar property.
2. The methods used to prove these theorems could all be extended to other geometric quantities.

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Thank you for listening!