Excursion sets of Planar Gaussian Fields

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Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^{3+}, stationary Gaussian field with mean zero, variance one, covariance function $K : \mathbb{R}^2 \to [-1, 1]$ and spectral measure ν, i.e. for $x, y \in \mathbb{R}^2$

$$K(x) = \mathbb{E}(f(y)f(y+x)) = \int_{\mathbb{R}^2} e^{it\cdot x} d\nu(t).$$

We are interested in the excursion sets

$$\left\{x \in \mathbb{R}^2 \;\middle|\; f(x) \geq \ell\right\}$$

and level sets

$$\left\{x \in \mathbb{R}^2 \;\middle|\; f(x) = \ell\right\}$$

restricted to a large domain, for $\ell \in \mathbb{R}$.

1) Applications

- Statistical testing in cosmology
- Nodal sets of Laplace eigenfunctions in Quantum Chaos
- Statistical version of Hilbert's 16-th problem (on 'typical' real algebraic hypersurfaces)

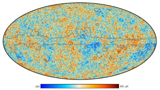
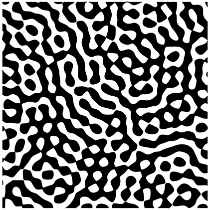


Figure: Fluctuations of the Cosmic Microwave Background Radiation (CMBR) (Source: Planck 2018).

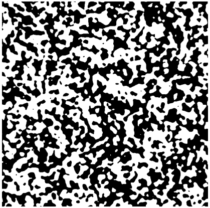
2) Connection to percolation theory

- Gaussian fields are predicted to behave analogously to discrete percolation models (Bogomolny-Schmit conjecture)
- Percolation results for fields have recently been proven (including the phase transition)
- We would like to prove/understand this 'universality'

1. Random Plane wave
   - $K(x) = J_0(|x|)$ the 0-th Bessel function
   - ν is normalised Lebesgue measure on the unit circle
   - Realisations of f are eigenfunctions of the Laplacian with eigenvalue -1
2. Bargmann-Fock field
   - $K(x) = \exp(-|x|^2/2)$
   - $\nu(t) = \exp(-|t|^2/2)\, dt$
   - scaling limit of random homogeneous polynomials on $\mathbb{R}P^2$



(a) Nodal set (i.e. zero level set) of Random Plane Wave



(b) Nodal set of Bargmann-Fock field

Let $N_{ES}(\ell, R)$ be the number of components of $\{f \geq \ell\}$ in $B(R)$ and $N_{LS}(\ell, R)$ the corresponding number for $\{f = \ell\}$.

- Main quantity of interest in my thesis
- Difficult to study due to non-locality

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Theorem (Nazarov-Sodin 2016, Kurlberg-Wigman 2017)

There exists $c_{ES}(\ell) \geq 0$ such that

$$\mathbb{E}(N_{ES}(\ell, R)) = c_{ES}(\ell) \cdot \pi R^2 + O(R) \qquad \text{as } R \to \infty.$$

If f is ergodic then

$$N_{ES}(\ell, R)/(\pi R^2) \to c_{ES}(\ell)$$

a.s. and in L^1 as $R \to \infty$.
The same result holds if N_{ES} and c_{ES} are replaced by N_{LS} and c_{LS} respectively.

- The proof relies on an ergodic argument and the fact that the number of components is 'semi-local'.

First main result
Poincaré-Hopf type theorem

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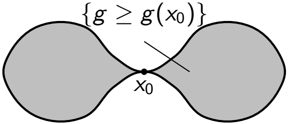
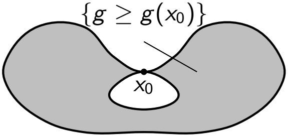
Definition Let $g : \mathbb{R}^2 \to \mathbb{R}$ have a saddle point x_0 (and no other critical points at the same level). Then x_0 is *lower connected* if it is in the closure of only one component of $\{g < g(x_0)\}$. Similarly, x_0 is *upper connected* if it is in the closure of only one component of $\{g > g(x_0)\}$.



(a) Lower connected



(b) Upper connected

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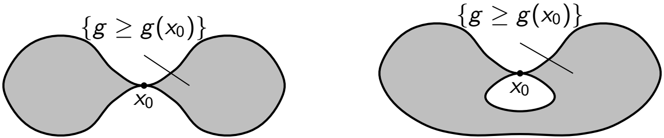

(a) Lower connected (b) Upper connected

Proposition For f a C^3 stationary Gaussian field there exists a function p_{s-} such that

$$\mathbb{E}\left(\#\{\text{Lower connected saddles in } B(R) \text{ at height} \geq \ell\}\right) = \pi R^2 \int_\ell^\infty p_{s-}(x)\, dx$$

There exist corresponding densities $p_{s^+}, p_s, p_{m^+}, p_{m-}$ for upper connected saddles, saddles, local maxima and local minima.

Theorem

Let f be a C^{3+} stationary Gaussian field satisfying some non-degeneracy assumptions, then

$$c_{ES}(\nu,\ell) = \int_{\ell}^{\infty} p_{m^+}(x) - p_{s^-}(x)\,dx$$

$$c_{LS}(\nu,\ell) = \int_{\ell}^{\infty} p_{m^+}(x) - p_{s^-}(x) + p_{s^+}(x) - p_{m^-}(x)\,dx.$$

Hence c_{LS} and c_{ES} are absolutely continuous in ℓ. In addition c_{LS} and c_{ES} are jointly continuous in (ν,ℓ) provided ν has a fixed compact support.

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Hence c_{LS} and c_{ES} are absolutely continuous in ℓ. In addition c_{LS} and c_{ES} are jointly continuous in (ν,ℓ) provided ν has a fixed compact support.

- This result essentially relies on a deterministic decomposition of excursion sets into critical points
- The same decomposition is applied to a simple example to explicitly derive c_{ES} and c_{LS}

Second main result
Smoothness/monotonicity

Theorem

Let f be a C^3 stationary Gaussian field such that

1. $\max_{\alpha \leq 3} |\partial^\alpha K(x)| \leq c|x|^{-(1+\epsilon)}$
2. $\mathbb{P}(f \in Arm_0(r, R)) \leq c_1 (r/R)^{c_2}$
3. *the spectral measure has a density around the origin bounded away from* 0

then $c_{ES}(\ell)$ and $c_{LS}(\ell)$ are continuously differentiable.

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- The one arm decay has been proven elsewhere assuming K is integrable
- This result holds for Bargmann-Fock but not (as written) for the Random Plane Wave
- Differentiability actually follows from

$$\mathbb{P}\left(\tilde{f}_\ell \text{ had an infinite four-arm saddle}\right) = 0$$

where $\tilde{f}_\ell$ is the field conditioned to have a saddle point at the origin.

Theorem

Let f be a Gaussian field satisfying the previous assumptions and some further non-degeneracy conditions, or let f be the Random Plane Wave, then $p_{s-}(\ell)/p_s(\ell)$ is non-decreasing in ℓ.

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Theorem

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Corollary

For the Random Plane Wave

$$D_+ c_{ES}(\ell) > 0 \quad \text{if } \ell \in (-\infty, 0.876]$$
$$D^+ c_{ES}(\ell) < 0 \quad \text{if } \ell \in [1, \infty)$$

where D_+, D^+ denote lower and upper Dini-derivatives. For the Bargmann-Fock field there exists $\epsilon > 0$ such that

$$c'_{ES}(\ell) \begin{cases} > 0 & \text{for } \ell \in [-\epsilon, 0.64] \\ < 0 & \text{for } \ell \in [1.02, \infty) \end{cases}$$

- The proof is to show that $\tilde{f}_\ell - \ell$ is stochastically decreasing in ℓ.

Theorem

Let f be a Gaussian field satisfying the previous assumptions (one arm decay, covariance decay etc) and suppose that $c_{ES}'(\ell) \neq 0$ then for some $c > 0$

$$\mathrm{Var}(N_{ES}(R,\ell)) \geq cR^2.$$

Theorem

Let f be the Random Plane Wave and $\ell \neq 0$, if $D_+ c_{ES}(\ell) > 0$ (or $D^+ c_{ES}(\ell) < 0$) then for some $c > 0$

$$\mathrm{Var}(N_{ES}(\ell,R)) \geq cR^3.$$

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Theorem

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Let f be the Random Plane Wave and $\ell \neq 0$, if $D_+ c_{ES}(\ell) > 0$ (or $D^+ c_{ES}(\ell) < 0$) then for some $c > 0$

$$\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^3.$$

- These bounds should be sharp for general fields/levels.
- We obtain intermediate variance bounds for fields with spectral blowup at the origin.
- The same results hold if N_{ES} and c_{ES} are replaced by N_{LS} and c_{LS}.

Corollary

For the Bargmann-Fock field:

- *if* $\ell \in (-\epsilon, 0.64) \cup (1.02, \infty)$ *then* $\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^2$,
- *if* $|\ell| > 1.37$ *then* $\mathrm{Var}(N_{LS}(\ell, R)) \geq cR^2$.

Corollary

For the Random Plane Wave

- *if* $\ell \in (-\infty, 0) \cup (0, 0.87) \cup [1, \infty)$ *then* $\mathrm{Var}(N_{ES}(\ell, R)) \geq cR^3$,
- *if* $\ell \in (-\infty, -1] \cup [1, \infty)$ *then* $\mathrm{Var}(N_{LS}(\ell, R)) \geq cR^3$.

Third main result

Variance lower bounds

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The overall method uses an elementary lemma due to Chatterjee: it is sufficient to show that as $R \to \infty$

1. $N_{ES}(\ell, R) - N_{ES}(\ell + a_R, R)$ fluctuates with order $R^2 a_R$
2. $d_{TV}(N_{ES}(\ell, R), N_{ES}(\ell + a_R, R)) \to 0$

where $a_R \to 0$ at a rate depending on the field.

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where $a_R \to 0$ at a rate depending on the field.

- To show the first point, we apply the second moment method, using the fact that the derivative of c_{ES} is bounded away from zero. This relies heavily on the analysis in earlier parts of the thesis.
- For the second point;

  $$d_{TV}(N_{ES}(\ell, R), N_{ES}(\ell + a_R, R)) \leq d_{TV}(f, f - a_R)$$

  and bound the latter quantity.
- For general fields, we use an abstract Cameron-Martin argument. For the Random Plane Wave we work with an explicit orthogonal expansion.