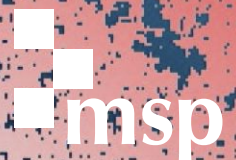


# PROBABILITY and MATHEMATICAL PHYSICS

## FLUCTUATIONS OF THE NUMBER OF EXCURSION SETS OF PLANAR GAUSSIAN FIELDS

DMITRY BELIAEV, MICHAEL MCAULEY AND STEPHEN MUIRHEAD

3:1  
2022



## FLUCTUATIONS OF THE NUMBER OF EXCURSION SETS OF PLANAR GAUSSIAN FIELDS

DMITRY BELIAEV, MICHAEL MCAULEY AND STEPHEN MUIRHEAD

For a smooth, stationary, planar Gaussian field, we consider the number of connected components of its excursion set (or level set) contained in a large square of area  $R^2$ . The mean number of components is known to be of order  $R^2$  for generic fields and all levels. We show that for certain fields with positive spectral density near the origin (including the Bargmann–Fock field), and for certain levels  $\ell$ , these random variables have fluctuations of order at least  $R$ , and hence variance of order at least  $R^2$ . In particular this holds for excursion sets when  $\ell$  is in some neighbourhood of zero, and it holds for excursion/level sets when  $\ell$  is sufficiently large. We prove stronger fluctuation lower bounds of order  $R^\alpha$  for  $\alpha \in [1, 2]$  in the case that the spectral density has a singularity at the origin. Finally we show that the number of excursion/level sets for the random plane wave at certain levels has fluctuations of order at least  $R^{3/2}$ , and hence variance of order at least  $R^3$ . We expect that these bounds are of the correct order, at least for generic levels.

### 1. Introduction

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous, centred stationary Gaussian field. We study the (upper) excursion sets and level sets of  $f$ , that is, the random sets

$$\{f \geq \ell\} := \{x \in \mathbb{R}^2 \mid f(x) \geq \ell\} \quad \text{and} \quad \{f = \ell\} := \{x \in \mathbb{R}^2 \mid f(x) = \ell\}$$

for  $\ell \in \mathbb{R}$ . For a wide class of fields, and for many levels  $\ell$ , we derive lower bounds on the fluctuations of the number of connected components of these sets contained inside large domains. We expect that these bounds are of the correct order for all levels to which our results apply, and for all but finitely many levels. However our results do not apply to the number of level sets at the nodal level  $\ell = 0$ , which may have lower order fluctuations for some fields (see [Section 2B1](#)).

Gaussian fields are used as a model for spatial phenomena in many fields of science (e.g., in quantum chaos [\[22\]](#), medical imaging [\[50\]](#), oceanography [\[3\]](#), cosmology [\[4\]](#), etc.), and the analysis of their excursion/level sets has many potential applications [\[3; 4; 22; 40; 50\]](#). To give an example, cosmological theories predict that the cosmic microwave background radiation (CMBR) can be modelled as a realisation of an isotropic Gaussian field on the two-dimensional sphere [\[4\]](#). One way to test this prediction is to compare geometric properties of the excursion/level sets of the CMBR with the Gaussian model; for instance, a recent analysis [\[40\]](#) used the number of excursion set components as a test statistic. We expect that a rigorous understanding of the fluctuations of this quantity will make such statistical analyses more robust.

*MSC2020:* 58K05, 60G15, 60G60.

*Keywords:* Gaussian fields, level sets, excursion sets, nodal sets, fluctuations, variance bounds.

The number of connected components of the excursion/level sets of a Gaussian field are inherently difficult quantities to study because they are “nonlocal”; the number of components in a domain cannot be counted by partitioning the domain and summing the number of components in each subdomain, since some components will intersect multiple subdomains. This can be contrasted with other “local” functionals, such as the length of a level set [30], the number of critical points [38], or the Euler characteristic of an excursion set [25; 34] (the locality of which can be seen from the Gauss–Bonnet theorem).

Functionals of Gaussian fields which are “nonlocal” cannot easily be analysed using classical tools such as the Kac–Rice formula [1, Chapter 11] or the Wiener chaos expansion [24; 39]. Nevertheless, the number of excursion/level set components of planar Gaussian fields has recently been studied using other, more general, techniques. Nazarov and Sodin [36] used an ergodic argument to prove a law of large numbers. Specifically, they showed that if  $f$  is an ergodic field satisfying some regularity assumptions,  $D_R := (-R/2, R/2)^2$  is the open square of side length  $R$  centred at the origin, and  $N_{\text{LS}}(D_R, \ell)$  denotes the number of components of the level set  $\{f = \ell\}$  contained in  $D_R$  (i.e., those which intersect  $D_R$  but not  $\partial D_R$ ), then there exists a constant  $c_{\text{LS}} > 0$  such that

$$\frac{N_{\text{LS}}(D_R, 0)}{R^2} \rightarrow c_{\text{LS}}$$

as  $R \rightarrow \infty$ , where convergence occurs in  $L^1$  and almost surely. Although this result was stated only for the nodal set (i.e., the zero level set), the arguments in [36] go through verbatim for excursion/level sets at arbitrary levels.

Results on the fluctuations of the number of excursion/level set components are comparatively lacking. Each excursion set component contains at least one critical point, and each level set component “surrounds” an (upper or lower) excursion set component. Since the number of critical points in a domain has a finite second moment which scales like the square of the area of the domain [17; 18], it follows that there exists a positive constant  $c_1 = c_1(\ell)$  such that, for all sufficiently large  $R$ ,

$$\text{Var}(N_{\text{LS}}(D_R, \ell)) < c_1 R^4 \quad \text{and} \quad \text{Var}(N_{\text{ES}}(D_R, \ell)) < c_1 R^4, \tag{1-1}$$

where  $N_{\text{ES}}(D_R, \ell)$  denotes the number of components of  $\{f \geq \ell\}$  contained in  $D_R$ . While the upper bound of order  $R^4$  is attained in certain degenerate cases (see Proposition 2.19), it is expected that the number of excursion/level sets of generic fields (i.e., those with rapid correlation decay) has variance of order exactly  $R^2$  (see Section 2B).

To the best of our knowledge, up until now the only nontrivial lower bound on the variance of either  $N_{\text{LS}}(D_R, \ell)$  or  $N_{\text{ES}}(D_R, \ell)$  is the recent result of Nazarov and Sodin [37] that  $\text{Var}(N_{\text{LS}}(D_R, 0))$  grows at least like some positive power of  $R$  (more precisely, they consider a related model of sequences of Gaussian fields on the sphere); the exponent in their bound is unspecified and not expected to be optimal. Their methods would likely extend to proving similar bounds for level sets at nonzero levels or excursion sets.

Nazarov and Sodin [35] have also proven the nontrivial upper bound  $\text{Var}(N_{\text{LS}}(D_R, 0)) < cR^{4-2/15}$  in the case of random spherical harmonics (this is an ensemble of Gaussian fields on the sphere which have the random plane wave as their local scaling limit [49]; we consider this model in detail below). This bound is a byproduct of an exponential concentration result. A slightly weaker concentration result was

subsequently proven for the random plane wave [41], implying the bound  $\text{Var}(N_{\text{LS}}(D_R, 0)) < cR^{4-1/8}$  for this model. Weaker concentration bounds have also been established for general fields [8; 43].

We prove lower bounds on  $\text{Var}(N_{\text{LS}}(D_R, \ell))$  and  $\text{Var}(N_{\text{ES}}(D_R, \ell))$  that are, conjecturally at least, of the correct order. To summarise our main results (see Theorems 2.7 and 2.15), we show that for a wide class of Gaussian fields there exists an exponent  $\alpha \in [2, 4]$  such that, for many levels  $\ell$ ,

$$\text{Var}(N_{\text{LS}}(D_R, \ell)) > cR^\alpha \quad \text{and} \quad \text{Var}(N_{\text{ES}}(D_R, \ell)) > cR^\alpha \tag{1-2}$$

for some  $c = c(\ell) > 0$  and all  $R$  sufficiently large. The value of  $\alpha \in [2, 4]$  is explicit and depends on the behaviour at the origin of the spectral measure of the field (see (2-4) for the definition of the spectral measure). For fields with rapid correlation decay and positive spectral density at the origin, the bound (1-2) holds for  $\alpha = 2$ , whereas for fields whose spectral measure has a singularity at the origin, (1-2) holds for an  $\alpha \in (2, 4)$  that depends on the polynomial exponent of the singularity. We also study the important special case of the random plane wave, for which we show that the bound (1-2) holds for  $\alpha = 3$ .

Bogomolny and Schmit [11] conjectured that the number of nodal domains for the random plane wave can be modelled by critical bond percolation on  $\mathbb{Z}^2$ . The generalised Bogomolny–Schmit conjecture states that for a wide class of Gaussian fields their nodal sets and general excursion sets can be modelled by critical and correspondingly off-critical percolation. Interestingly, our result is inconsistent with the generalised Bogomolny–Schmit conjecture for the random plane wave and nonzero levels [12] (see the discussion in Section 2B), although our results do not apply to the nodal set, which is the most important case of the conjecture.

We establish the variance bounds in (1-2) for a wide range of levels. For general fields, the bound for excursion sets holds for all levels  $\ell$  in a neighbourhood of zero (the nodal level  $\ell = 0$  is excluded for the random plane wave), and when  $\ell$  is sufficiently large the bound holds for both excursion and level sets (see Corollaries 2.12, 2.14 and 2.17). Indeed, Theorems 2.7 and 2.15 give a sufficient condition for (1-2) which we expect to be satisfied for all but a very small, finite number of levels  $\ell$ . In fact we suspect (see Section 2B) that this condition should fail for only one or three values of  $\ell$  (depending on the field).

On the other hand, we do not expect that (1-2) is necessarily true for all levels. While we conjecture that (1-2) holds for generic levels, we expect that for some fields there exists a finite set of “anomalous” levels at which the variance is of strictly lower order (see Conjectures 2.21 and 2.22 for a precise statement). This phenomenon is reminiscent of “Berry cancellation”, i.e., the known fact that, for some fields, such as the random plane wave, the variance of the length of the nodal set is of lower order than for nonzero levels [10; 39; 48].

## 2. Main results

We consider a Gaussian field  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  which is continuous, centred and stationary, and let  $\kappa(x) := \mathbb{E}(f(x)f(0))$  be its covariance function. Throughout the paper we make the following basic assumption:

**Assumption 2.1.** The Gaussian field  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $C^3$ -smooth almost surely, and normalised so that, for each  $x \in \mathbb{R}^2$ ,

$$\mathbb{E}(f(x)) = 0, \quad \text{Var}(f(x)) = 1 \quad \text{and} \quad \text{Cov}(\nabla f(x)) = cI_2, \tag{2-1}$$

where  $c$  is a positive constant and  $I_2$  is the  $2 \times 2$  identity matrix. In addition we assume that

$$\max_{|\alpha| \leq 2} |\partial_\alpha \kappa(x)| \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

and that, for every  $x \in \mathbb{R}^2 \setminus \{0\}$ ,

$$\nabla^2 f(0) \quad \text{and} \quad (f(x), f(0), \nabla f(x), \nabla f(0)) \tag{2-2}$$

are nondegenerate multivariate Gaussian random variables.

This assumption implies, in particular, that  $\kappa$  is of class  $C^6$  [36, Appendix A.3], and also that the field is ergodic [36, Appendix B]. We impose the normalisation (2-1) for simplicity; since  $(f(0), \nabla f(0))$  is assumed to be nondegenerate we can always apply a linear rescaling and rotation to the domain of  $f$  so that (2-1) holds. A sufficient condition for (2-2) to be nondegenerate is that the support of the spectral measure (see the definition in (2-4)) contains either an open set or a centred ellipse [7, Lemma A.2].

We begin by formally stating the law of large numbers for excursion/level sets (noting that this actually holds under weaker conditions than those which we give). We fix an open rectangle  $D \subset \mathbb{R}^2$  centred at the origin. For  $R \geq 1$ , we let  $D_R = \{x \in \mathbb{R}^2 : x/R \in D\}$  and let  $N_{\text{ES}}(D_R, \ell)$  denote the number of components of  $\{f \geq \ell\}$  contained in  $D_R$  (i.e., those which intersect  $D_R$  but not  $\partial D_R$ ). We define  $N_{\text{LS}}(D_R, \ell)$  analogously for  $\{f = \ell\}$ .

**Theorem 2.2** [6; 26; 36]. *Let  $f$  be a Gaussian field satisfying Assumption 2.1. For each  $\ell \in \mathbb{R}$ , there exist  $c_{\text{ES}}(\ell), c_{\text{LS}}(\ell) \geq 0$  such that*

$$\begin{aligned} \mathbb{E}(N_{\text{ES}}(D_R, \ell)) &= c_{\text{ES}}(\ell) \cdot \text{Area}(D) \cdot R^2 + O(R), \\ \mathbb{E}(N_{\text{LS}}(D_R, \ell)) &= c_{\text{LS}}(\ell) \cdot \text{Area}(D) \cdot R^2 + O(R) \end{aligned}$$

as  $R \rightarrow \infty$ . The constants implied by the  $O(\cdot)$  notation are independent of  $\ell$ . Furthermore

$$\frac{N_{\text{ES}}(D_R, \ell)}{\text{Area}(D) \cdot R^2} \rightarrow c_{\text{ES}}(\ell) \quad \text{and} \quad \frac{N_{\text{LS}}(D_R, \ell)}{\text{Area}(D) \cdot R^2} \rightarrow c_{\text{LS}}(\ell)$$

almost surely and in  $L^1$ .

The limiting constants  $c_{\text{ES}}(\ell)$  and  $c_{\text{LS}}(\ell)$  describe the asymptotic density of excursion sets and level sets respectively. Since they are defined implicitly, very little is known rigorously about them. In [6] a representation was given in terms of critical points of various types. For  $R > 0$  and  $a \leq b$ , we define  $N_h(D_R, [a, b])$ , for  $h = m^+, m^-, s^+, s^-$ , to be the number of local maxima, local minima, upper connected saddles and lower connected saddles respectively of  $f$  in  $D_R$  with level in  $[a, b]$  (see [6] for the definition of upper/lower connected saddles; the precise definition has no relevance to the current paper).

**Theorem 2.3** [6, Proposition 1.8, Theorem 1.9]. *Let  $f$  be a Gaussian field satisfying Assumption 2.1, and let  $D$  be an open rectangle centred at the origin. Then for all  $R > 0$  and  $a \leq b$ ,*

$$c_{\text{ES}}(a) - c_{\text{ES}}(b) = \frac{1}{\text{Area}(D)R^2} \left( \mathbb{E}(N_{m^+}(D_R, [a, b]) - N_{s^-}(D_R, [a, b])) \right),$$

and

$$c_{\text{LS}}(a) - c_{\text{LS}}(b) = \frac{1}{\text{Area}(D)R^2} (\mathbb{E}(N_{m^+}(D_R, [a, b]) - N_{s^-}(D_R, [a, b])) + \mathbb{E}(N_{s^+}(D_R, [a, b]) - N_{m^-}(D_R, [a, b]))).$$

It can be deduced from the above representation that  $c_{\text{ES}}$  and  $c_{\text{LS}}$  are absolutely continuous. In [7] additional smoothness and monotonicity properties of  $c_{\text{ES}}$  and  $c_{\text{LS}}$  were derived; for instance, it was shown that  $c_{\text{ES}}(\ell)$  and  $c_{\text{LS}}(\ell)$  are continuously differentiable in  $\ell$  for a wide class of fields.

**2A. Fluctuations of the number of level/excursion set components.** Our main results concern the order of fluctuations of  $N_{\text{ES}}$  and  $N_{\text{LS}}$ . To formalise this concept we use the following definition, taken from [15].

**Definition 2.4.** Let  $X_n$  be a sequence of random variables and  $u_n$  a sequence of positive real numbers. We say that  $X_n$  has *fluctuations of order at least  $u_n$*  if there exist  $c_1, c_2 > 0$  such that, for all sufficiently large  $n$  and all real numbers  $a \leq b$  with  $b - a \leq c_1 u_n$ ,

$$\mathbb{P}(a \leq X_n \leq b) \leq 1 - c_2.$$

Similarly, we say that a collection of random variables  $(X_R)_{R \geq 0}$  has *fluctuations of order at least  $(u_R)_{R \geq 0}$*  if, for any increasing sequence  $R_n \rightarrow \infty$ , the sequence of random variables  $X_{R_n}$  has fluctuations of order at least  $u_{R_n}$ .

It is easy to see that if a collection of random variables  $(X_n)_{n \geq 0}$  has fluctuations of order at least  $(u_n)_{n \geq 0}$  then it has variance of order at least  $u_n^2$ , i.e., there exists  $c > 0$  such that

$$\text{Var}(X_n) > c u_n^2 \tag{2-3}$$

for all  $n$  sufficiently large. On the other hand, having fluctuations of order at least  $u_n$  is a strictly stronger statement than (2-3), since the latter is consistent with the bulk of the probability mass concentrating on arbitrarily small scales.

We now present our main results on the fluctuations of  $N_{\text{ES}}$  and  $N_{\text{LS}}$ , which are divided into three statements. The first applies to general fields, and in particular to fields that have either (i) fast correlation decay and positive spectral density at the origin or (ii) spectral measure with a singularity at the origin. The second concerns the special case of the random plane wave. The third treats a certain class of somewhat degenerate fields.

**2A1. General fields.** To state our first result we introduce some additional assumptions on the field  $f$ . Recall that  $\kappa$  is the covariance function of  $f$ . Since  $\kappa$  is continuous, Bochner’s theorem states that it is the Fourier transform of a positive measure  $\mu$  which is Hermitian (i.e.,  $\mu(A) = \mu(-A)$  for any Borel set  $A$ ); that is, for all  $x \in \mathbb{R}^2$ ,

$$\kappa(x) = \int_{\mathbb{R}^2} e^{2\pi i t \cdot x} d\mu(t). \tag{2-4}$$

We refer to  $\mu$  as the *spectral measure* of the field. For some of our results we will assume that  $\mu$  has a density; provided it exists, we denote this by  $\rho(\cdot)$ . Under [Assumption 2.1](#) the spectral measure is a probability measure.

**Assumption 2.5.** There exists a neighbourhood  $V \subset \mathbb{R}^2$  of the origin such that the spectral measure  $\mu$  has density  $\rho$  on  $V$  and  $\inf_V \rho > 0$ .

The simplest way to guarantee the existence of  $\rho$  is to assume that  $\kappa \in L^1$ ; in this case  $\rho$  is uniformly continuous. If we additionally assume that  $\int \kappa = \rho(0) > 0$ , then there is a neighbourhood of the origin where  $\rho$  is bounded away from 0. On the other hand, [Assumption 2.5](#) also allows for  $\rho$  to have a singularity at the origin.

Under [Assumption 2.5](#) the support of  $\mu$  contains an open set, and so the Gaussian vector formed from  $f, \nabla f$  and  $\nabla^2 f$  at a finite number of distinct points is nondegenerate (see [7, Lemma A.2]).

For the case in which the spectral measure does not have a singularity at the origin, we will need to assume some extra control over the saddle points of the field. Let  $x_0 \in \mathbb{R}^2$  be a saddle point of a function  $g \in C_{\text{loc}}^2(\mathbb{R}^2)$  such that  $g$  has no other critical points at the same level as  $x_0$ . We say that  $x_0$  is *four-arm* in  $D_R$  if it is in the closure of two components of  $\{x \in D_R : g(x) > g(x_0)\}$  and two components of  $\{x \in D_R : g(x) < g(x_0)\}$ . Under some regularity assumptions on the function  $g$ , this definition implies that the four level lines (or ‘‘arms’’) which ‘‘emerge’’ from  $x_0$  will all hit the boundary of  $D_R$  (i.e., they will not ‘‘join together’’). For  $a \leq b$ , let  $N_{4\text{-arm}}(D_R, [a, b])$  be the number of saddle points of  $f$  which are four-arm in  $D_R$  and have level in  $[a, b]$ .

**Assumption 2.6.** For each open rectangle  $D$  centred at the origin and every  $a \leq b$ , there exists a function  $\delta_R \rightarrow 0$  as  $R \rightarrow \infty$  and a constant  $c > 0$  such that, for each  $R > 1$  and  $a \leq a_R \leq b_R \leq b$ ,

$$\mathbb{E}(N_{4\text{-arm}}(D_R, [a_R, b_R])) \leq c \min\{\delta_R R^2(b_R - a_R), R\}.$$

Sufficient conditions for a field to satisfy [Assumption 2.6](#) are given in [7, Corollary 2.12] (this result actually gives the analogous bound for the expected number of four-arm saddle points in  $B(R)$  the ball of radius  $R$ , but since  $B(c_D R) \subset D_R \subset B(C_D R)$  for some constants  $c_D, C_D > 0$  and all  $R > 0$ , the two statements are equivalent). In particular this assumption is satisfied for isotropic fields whose correlations are positive and rapidly decaying, which includes the important special case of the Bargmann–Fock field, i.e., the field with covariance function  $\kappa(x) = \exp(-|x|^2/2)$  (see [5] for background).

We can now state our fluctuation lower bound for general fields. Recall that the *Dini derivatives* are a generalisation of the usual derivative, and coincide in the case of continuously differentiable functions (see (3-1) and (3-2) for a formal definition).

**Theorem 2.7.** *Let  $f$  be a Gaussian field satisfying Assumptions 2.1 and 2.5. Define  $g(r) = \inf_{x \in B(2r)} \rho(x)$ . Let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin and recall that  $D_R = \{x \in \mathbb{R}^2 : x/R \in D\}$ . Suppose further that at least one of the following holds:*

- (1) *the field  $f$  satisfies [Assumption 2.6](#), or*
- (2) *the spectral measure  $\mu$  has a singularity at the origin, i.e.,  $g(r) \rightarrow \infty$  as  $r \rightarrow 0$ .*

*If  $c_{\text{ES}}$  has a positive lower Dini derivative at  $\ell$  (or a negative upper Dini derivative), then  $(N_{\text{ES}}(D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R\sqrt{g(1/R)}$ , and hence variance of order at least  $R^2 g(1/R)$ . The same conclusion holds if we replace  $N_{\text{ES}}$  and  $c_{\text{ES}}$  with  $N_{\text{LS}}$  and  $c_{\text{LS}}$  respectively.*

**Remark 2.8.** The variance lower bound  $R^2g(1/R)$  interpolates between  $R^2$  (if the spectral density is bounded at the origin) and  $o(R^4)$  (note that  $g(1/R) = o(R^2)$  since  $\rho$  is integrable on a neighbourhood of the origin). This is consistent with the trivial upper bound in (1-1).

**Remark 2.9.** It is shown in [7] that  $c_{ES}$  and  $c_{LS}$  are continuously differentiable for a wide class of Gaussian fields, and in this case the conditions on Dini derivatives in Theorem 2.7 are equivalent to the conditions  $c'_{ES}(\ell) \neq 0$  and  $c'_{LS}(\ell) \neq 0$ . We expect that  $c_{ES}$  and  $c_{LS}$  are continuously differentiable in general, but we lack a comprehensive proof.

For general fields we expect that  $c_{ES}$  and  $c_{LS}$  have nonzero derivative for all but a small finite number of levels  $\ell$  (see Section 2B). In fact, based on simulations (see [40]) we expect  $c_{ES}$  to be unimodal. We also suspect that  $c_{LS}$  should be either unimodal or bimodal depending on the field; in [6] we describe a somewhat degenerate class of fields for which this is true and we show that for certain more interesting fields, including the random plane wave,  $c_{LS}$  is at least bimodal (i.e., has at least two local maxima). We therefore hope that Theorem 2.7 can eventually be applied to all but a finite number of levels. On the other hand, Theorem 2.7 cannot be applied directly to  $N_{LS}(D_R, 0)$ , since by symmetry  $c'_{LS}(0) = 0$  whenever the derivative is defined.

In Section 2B we give some motivation for why  $c'_{ES}(\ell) \neq 0$  and  $c'_{LS}(\ell) \neq 0$  are, in a sense, natural conditions for a lower bound on fluctuations.

**Remark 2.10.** The case of spectral singularity ( $g(r) \rightarrow \infty$ ) is closely related to the case of *long-range dependence*, i.e., the case in which  $\kappa$  decays sufficiently slowly so as not to be integrable. In particular, standard Abel/Tauberian theorems [27, Chapter 1.4] imply that, up to some regularity assumptions, the asymptotics  $\rho(x) \sim |x|^{-\alpha}$  as  $|x| \rightarrow 0$  and  $\kappa(x) \sim |x|^{\alpha-2}$  as  $|x| \rightarrow \infty$  are equivalent for  $\alpha \in (0, 2)$ . Hence, broadly speaking, our results shows that if correlations decay polynomially with exponent  $\beta \in (0, 2)$ , then the variance of  $N_{ES}$  and  $N_{LS}$  grow at order at least  $R^{4-\beta}$ . This is analogous to known results on fluctuations of “local” functionals of long-range dependent Gaussian processes and fields [27; 45].

**Remark 2.11.** Recall that  $N_{ES}(D_R, \ell)$  and  $N_{LS}(D_R, \ell)$  count the number of connected components of the excursion/level sets that intersect  $D_R$  but which do not intersect the boundary  $\partial D_R$ ; a natural question is whether the result still holds if we include components which intersect the boundary (either with or without multiplicity for repeated intersections). Since the trivial upper bound on the second moment of boundary components is  $O(R^2)$ , this is immediate in cases in which the variance bound is of order exceeding  $R^2$ . While in the general case it does not follow from our stated results, our proofs can easily be adapted to cover boundary components, but for brevity we omit the details.

In order to extract from Theorem 2.7 a concrete statement about the fluctuations of  $N_{ES}$  and  $N_{LS}$ , one needs to show that the (Dini) derivatives of  $c_{ES}$  and  $c_{LS}$  are nonzero for particular levels. In previous work [7] we proved monotonicity results for  $c_{ES}$  and  $c_{LS}$  implying that this condition holds for certain ranges of levels. We illustrate this with the Bargmann–Fock field.

**Corollary 2.12.** *Let  $f$  be the Bargmann–Fock field and  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. There exists  $\epsilon > 0$  (independent of  $D$ ) such that the following holds. If  $\ell \in (-\epsilon, 0.64) \cup (1.03, \infty)$  then  $(N_{ES}(D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R$  and hence variance of order at least  $R^2$ . If  $|\ell| > 1.03$  then  $(N_{LS}(D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R$  and hence variance of order at least  $R^2$ .*



*Proof.* Assumptions 2.1 and 2.5 are trivially satisfied for the Bargmann–Fock field, and [7, Corollary 2.12] states that Assumption 2.6 is also satisfied. The corollary then follows from [7, Proposition 2.21], which states that  $c'_{\text{ES}}(\ell) \neq 0$  and  $c'_{\text{LS}}(\ell) \neq 0$  for the respective levels given above.  $\square$

For general isotropic fields satisfying some additional conditions listed below (Assumption 2.13), it is shown in [7] that  $c_{\text{ES}}$  and  $c_{\text{LS}}$  are monotone for similar ranges of levels, and so we draw similar conclusions for such fields. To be a bit more precise, for each such field there exists  $\epsilon > 0$  such that the fluctuations of  $(N_{\text{ES}}(D_R, \ell))_{R \geq 0}$  are of order at least  $R$  for  $\ell \in (-\epsilon, C) \cup (\sqrt{2}/\chi, \infty)$  and the fluctuations of  $(N_{\text{LS}}(D_R, \ell))_{R \geq 0}$  are of order at least  $R$  for  $|\ell| > \sqrt{2}/\chi$ . Here  $\chi$ , given by (2-5) below, is a parameter controlling the distribution of critical points [16] and  $C$  is the positive root of an explicit but rather complicated equation involving  $\chi$ , the normal density function and cumulative density function.

**Assumption 2.13.** The field  $f$  satisfies the following:

- $f$  is isotropic (i.e., its law is invariant under rotations) and

$$\chi := \frac{-\sqrt{3}\partial^{(2,0)}\kappa(0)}{\sqrt{\partial^{(4,0)}\kappa(0)}} \geq 1. \quad (2-5)$$

- There exist  $c, \nu > 0$  such that, for all  $|x| \geq 1$ ,

$$\max_{|\alpha| \leq 3} |\partial^\alpha \kappa(x)| \leq c|x|^{-(1+\nu)}.$$

- The Gaussian vector  $(f(0), \nabla^2 f(0))$  is nondegenerate, and for all  $x \in \mathbb{R}^2$ ,

$$\mathbb{E}\left(f(x) \mid f(0) = 0, \nabla^2 f(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \geq 0, \quad (2-6)$$

$$\mathbb{E}\left(f(x) \mid f(0) = 1, \nabla^2 f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) \leq 1. \quad (2-7)$$

- For  $0 < r < R$ , let  $\text{Arm}_\ell(r, R)$  denote the “one-arm event” that there exists a component of  $\{f \geq \ell\}$  which intersects both  $\partial B(r)$  and  $\partial B(R)$ . Then there exist  $c_1, c_2 > 0$  such that for any  $1 < r < R$

$$\mathbb{P}(f \in \text{Arm}_0(r, R)) \leq c_1(r/R)^{c_2}. \quad (2-8)$$

We note that the one-arm decay condition in this assumption has been verified for a wide class of fields [33; 42] and is believed to hold even more generally. All other parts of this assumption can be verified directly using the covariance function. More detail on how to verify conditions (2-6) and (2-7) is given in [7]. We omit this detail here, since Assumption 2.13 will not be used in the current paper other than for the following corollary.

**Corollary 2.14.** *Let  $f$  satisfy Assumptions 2.1, 2.5 and 2.13. Let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. There exists  $\epsilon > 0$  and an explicit constant  $C > 0$  (both independent of  $D$ ) such that the following holds. If  $\ell \in (-\epsilon, C) \cup (2/\chi, \infty)$  then  $(N_{\text{ES}}(D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R$  and hence variance of order at least  $R^2$ . If  $|\ell| > 2/\chi$  then  $(N_{\text{LS}}(D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R$  and hence variance of order at least  $R^2$ .*

*Proof.* Under these assumptions, [7, Proposition 2.22] states that  $c_{\text{ES}}$  and  $c_{\text{LS}}$  are continuously differentiable and that

$$c'_{\text{ES}}(\ell) \begin{cases} > 0 & \text{for } \ell \in (-\epsilon, C), \\ < 0 & \text{for } \ell \in (\sqrt{2}/\chi, \infty) \end{cases} \quad \text{and} \quad c'_{\text{LS}}(\ell) < 0 \quad \text{for } \ell \in (\sqrt{2}/\chi, \infty)$$

for  $\epsilon, C > 0$  as above. Since  $c_{\text{LS}}$  is symmetric in  $\ell$ , the result then follows from [Theorem 2.7](#). □

The general expression for the constant  $C$  in this result is given in the proof of [7, Proposition 2.22]. The expression is straightforward to evaluate numerically for any particular field (in particular, it depends only on the first two derivatives of the covariance function at the origin).

**2A2. The random plane wave.** We now turn to the important special case of the random plane wave (RPW), i.e., the field with covariance function  $\kappa(x) = J_0(|x|)$ , where  $J_0$  is the 0-th Bessel function. The RPW has applications in quantum chaos as a model for high energy eigenfunctions of the Laplacian (see [9]) and the geometry of its excursion/level sets have been studied by many authors (see [11; 22]).

The RPW does not fall within the scope of [Theorem 2.7](#) since it does not satisfy [Assumption 2.5](#) (its spectral measure is supported on the unit circle). Nevertheless we can prove the following bound on fluctuations.

**Theorem 2.15.** *Let  $f$  be the random plane wave and let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. If  $c_{\text{ES}}$  has a positive lower Dini derivative at  $\ell \neq 0$  (or a negative upper Dini derivative), then  $(N_{\text{ES}}(D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R^{3/2}$ , and hence variance of order at least  $R^3$ . The same conclusion holds if we replace  $N_{\text{ES}}$  and  $c_{\text{ES}}$  with  $N_{\text{LS}}$  and  $c_{\text{LS}}$  respectively.*

**Remark 2.16.** The larger fluctuations of  $N_{\text{ES}}$  and  $N_{\text{LS}}$  for the RPW (order  $R^{3/2}$  compared to the generic  $R$ ) can be understood as a reflection of degeneracies in the RPW, which manifest in at least four ways. First, the spectral measure  $\mu$  is supported on a dimension-one subspace (the unit circle). Second, and directly related to the first, is that realisations of the RPW are solutions of the Helmholtz equation  $\Delta f = -f$ . Third, the RPW has long-range dependence, with correlations decaying only at rate  $1/\sqrt{|x|}$ . Fourth, when expanded in a particular orthogonal series (see (3-14)), only order  $R$  terms are required to specify the RPW in a ball of radius  $R$ , up to exponentially small error, compared to the generic order  $R^2$  terms for a planar field. In fact, this last property is what ultimately drives our proof of [Theorem 2.15](#).

As for the Bargmann–Fock field, in previous work we verified the Dini derivative condition for  $c_{\text{ES}}$  and  $c_{\text{LS}}$  in certain intervals (see [7, Proposition 2.20]). This leads to the following corollary.

**Corollary 2.17.** *Let  $f$  be the random plane wave and let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. If  $\ell \in (-\infty, 0) \cup (0, 0.87) \cup [1, \infty)$  then  $(N_{\text{ES}}(D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R^{3/2}$  and hence variance of order at least  $R^3$ . If  $|\ell| \geq 1$  then  $(N_{\text{LS}}(D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R^{3/2}$  and hence variance of order at least  $R^3$ .*

**2A3. Degenerate fields.** Finally we consider the class of fields whose spectral measure has a delta mass at the origin. In this case, we prove that the variance attains the order of the trivial upper bound in (1-1) for all levels.

**Assumption 2.18.** The stationary Gaussian field  $f$  has spectral measure  $\nu = \alpha\delta_0 + \nu^*$ , where  $\alpha > 0$ ,  $\delta_0$  is a delta-mass at the origin and  $\nu^*$  is a (positive) measure. If  $g$  is the Gaussian field with spectral measure  $\nu^*$  then  $g$  satisfies [Assumption 2.1](#).

Under this assumption, the field  $f$  is no longer normalised to have variance one at a point: instead  $\text{Var}(f(x)) = 1 + \alpha$  and  $\text{Var}(g(x)) = 1$  for all  $x \in \mathbb{R}^2$ . This is motivated by the fact that we can represent  $f$  as

$$f = g + \sqrt{\alpha}Z,$$

where  $Z$  is a standard Gaussian variable independent of  $g$ . This representation follows immediately from considering the covariance function of the field on the right. In order to analyse the level sets of  $f$ , we apply [Theorem 2.2](#) to  $g$  and consider the additional effect of shifting the overall level due to  $Z$ . It is therefore convenient to normalise  $g$  as in the rest of the paper. Of course, our results apply to any stationary Gaussian field with spectral mass at the origin: one simply has to rescale the variance to match the normalisation above.

**Proposition 2.19.** *Let  $f$  satisfy [Assumption 2.18](#) and let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. For each  $\ell \in \mathbb{R}$ ,  $(N_{\text{ES}}(f; D_R, \ell))_{R \geq 0}$  has fluctuations of order at least  $R^2$ . Moreover, there exist positive constants  $c_1(\ell)$  and  $c_2(\ell)$  (independent of  $D$ ) such that*

$$c_1 \text{Area}(D)^2 R^4 < \text{Var}(N_{\text{ES}}(f; D_R, \ell)) < c_2 \text{Area}(D)^2 R^4$$

for all  $R > 0$  sufficiently large. The same conclusions hold if we replace  $N_{\text{ES}}$  and  $c_{\text{ES}}$  with  $N_{\text{LS}}$  and  $c_{\text{LS}}$  respectively.

This result roughly says that adding a random independent level shift to any nondegenerate Gaussian field (which is equivalent to adding a delta mass to the spectral measure at the origin — see above) ensures that the number of excursion/level set components of the resulting field has variance of maximal order at all levels.

We note that this proposition makes no requirement on the derivative of the mean functional at a given level. Intuitively this holds because the variable  $Z$  can always shift the field  $g$  to levels at which the asymptotic density of excursion (or level) components differs. Although this result follows from the methods we utilise throughout the rest of the paper (to be described in [Section 2C](#)) we actually prove it using more elementary techniques. This result is therefore included primarily for completeness and comparison, rather than as a significant contribution.

**Remark 2.20.** There are other degenerate classes of fields for which the variance of  $N_{\text{ES}}$  and  $N_{\text{LS}}$  can be shown to be of maximal order  $R^4$ :

- (1) For fields which are doubly periodic (i.e., have spectral measure which is supported on a lattice), it is evident that  $N_{\text{ES}}$  and  $N_{\text{LS}}$  have variance of order  $R^4$  whenever the variance is finite and nondegenerate (known under minimal assumptions; see [Remark 3.6](#)).
- (2) For fields with spectral measure supported on exactly four or five points, the entire distributions of  $N_{\text{ES}}$  and  $N_{\text{LS}}$  can be explicitly calculated (see [\[6, Proposition 1.20\]](#) or [\[31, Proposition 2.1.11\]](#)). In this case the variances of  $N_{\text{ES}}$  and  $N_{\text{LS}}$  can also be shown to have order  $R^4$  whenever they are nondegenerate.

**2B. Further discussion and open questions.** In this section we discuss conjectures, open questions, and links to other models.

**2B1. Anomalous levels.** As mentioned above, we believe that the variance bounds in Theorems 2.7 and 2.15 are of the correct order for generic levels, with the possible exception of a finite set of “anomalous” levels, different for  $c_{\text{ES}}$  and  $c_{\text{LS}}$ , at which the variance is of lower order.

**Conjecture 2.21.** Suppose that  $f$  satisfies Assumptions 2.1, 2.5 and 2.6 (e.g., the Bargmann–Fock field) and  $D \subset \mathbb{R}^2$  is an open rectangle centred at the origin. Then for all  $\ell \in \mathbb{R}$  there exists  $c_{\text{var}}(\ell) > 0$  such that

$$\text{Var}(N_{\text{ES}}(D_R, \ell)) \sim c_{\text{var}}(\ell) \text{Area}(D)R^2,$$

and the same conclusion is true for  $N_{\text{LS}}(D_R, \ell)$ .

**Conjecture 2.22.** Suppose that  $f$  satisfies Assumptions 2.1 and 2.5, and assume that there exist  $\alpha \in (0, 2)$  and  $r_0 > 0$  such that

$$\rho(x) = |x|^{-\alpha}$$

for all  $|x| < r_0$ . Let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. Then there exists a (possibly empty) finite set  $\mathcal{L} \subset \mathbb{R}$  and  $c_{\text{var}}(\ell) > 0$  such that, for all  $\ell \notin \mathcal{L}$ ,

$$\text{Var}(N_{\text{ES}}(D_R, \ell)) \sim c_{\text{var}}(\ell) \text{Area}(D)^{(2+\alpha)/2} R^{2+\alpha},$$

whereas for all  $\ell \in \mathcal{L}$ ,

$$\text{Var}(N_{\text{ES}}(D_R, \ell)) \ll R^{2+\alpha},$$

and the same conclusion is true for  $N_{\text{LS}}(D_R, \ell)$  (with a different set  $\mathcal{L}$ ). If  $f$  is the RPW, then the same conclusion is true with  $2 + \alpha$  replaced with 3.

These conjectures are motivated by a comparison with the known behaviour of the variance of the *Minkowski functionals* of the excursion sets, namely the volume of the excursion set, the length of the level set, and the Euler characteristic of the excursion set (by Hadwiger’s theorem, these form a linear basis for the set of “local” functionals of the excursion sets that are isometrically invariant [1]). To illustrate this, let  $L(R, \ell)$  denote the length of the level set  $\{f = \ell\}$  contained within  $[-R, R]^2$ . It is known [25] that for fields with rapid correlation decay, there exists  $c_{\text{var}}(\ell) > 0$  such that

$$\text{Var}(L(R, \ell)) \sim c_{\text{var}}(\ell)R^2,$$

whereas for the RPW

$$\text{Var}(L(R, \ell)) \sim \begin{cases} c_{\text{var}}(\ell)R^3 & \text{for } \ell \neq 0, \\ c_{\text{var}}(0)R^2 \log R & \text{for } \ell = 0. \end{cases} \tag{2-9}$$

In other words, for the RPW,  $L(R, \ell)$  has variance of lower order at level  $\ell = 0$  compared to  $\ell \neq 0$ . This phenomenon was first predicted by Berry [10], and has since been proven rigorously [39; 48]. A similar phenomenon is also known to occur for the volume of the excursion sets and the Euler characteristic (see [14; 28]); in the latter case the variance reduction also occurs at certain nonzero levels.

The phenomenon of variance reduction can be understood as reflecting the fact that, for the RPW, the fluctuations of the Minkowski functionals are dominated by the second term in their Wiener chaos expansion, whose coefficient as a function of  $\ell$  happens to vanish at certain levels (see the discussion in [39]).

The same is also known to be true in the case of spectral singularity at the origin [27, Chapter 3]. In contrast, for fields with rapid correlation decay, many terms in the Wiener chaos expansion have fluctuations of leading order (see for instance [19]), and so one should not expect anomalous levels since that would require many coefficients to vanish simultaneously.

**2B2. Further questions.** Assuming that Conjectures 2.21 and 2.22 are correct, they give rise to a number of further questions. For simplicity we discuss only the case of the excursion sets, but the analogous questions can be asked of the level sets.

A first set of questions concerns the anomalous levels  $\mathcal{L}$  in the case of the RPW or fields with spectral singularity.

**Question 2.23.** (1) Is the set of anomalous levels  $\mathcal{L}$  nonempty? What is its cardinality?

(2) Let  $\mathcal{C}$  denote the set of critical points of the density functional  $c_{\text{ES}}$ . By Theorems 2.7 and 2.15, we know that  $\mathcal{L} \subseteq \mathcal{C} \cup \{0\}$  for the RPW, whereas  $\mathcal{L} \subseteq \mathcal{C}$  in the spectral singularity case. Are these inclusions strict?

(3) What is the order of  $\text{Var}(N_{\text{ES}}(D_R, \ell))$  for  $\ell \in \mathcal{L}$ ? Does it depend on the field and on the level? Is it always of order at least  $R^2$ ?

Based on simulations we expect that  $c_{\text{ES}}$  is unimodal for general fields, which would imply that  $|\mathcal{C}| = 1$  and so  $|\mathcal{L}| \leq 1$  (or  $|\mathcal{L}| \leq 2$  for the RPW). We also expect that  $c_{\text{LS}}$  is either unimodal or bimodal, depending on the field, which would imply that  $|\mathcal{L}| \leq 3$  for level sets.

A second question concerns the constants  $c_{\text{var}}(\ell)$  for generic levels  $\ell \notin \mathcal{L}$ . For the Minkowski functionals of the RPW, it is known that  $c_{\text{var}}(\ell)$  is related to the derivative of the first moment (i.e., density) functional  $c(\ell)$  via

$$c_{\text{var}}(\ell) \propto (\ell c'(\ell))^2; \tag{2-10}$$

see the formulas and discussion presented in [13; 14] (actually (2-10) has only been proven for the related model of the random spherical harmonics, but we expect it to hold also for the RPW). In particular, levels are anomalous precisely when either  $\ell = 0$  or  $c'(\ell) = 0$ , which are exactly the conditions for which our bound in Theorem 2.15 holds. This is evidence that our conditions in Theorem 2.15 are quite natural.

We are not aware of any similar results to (2-10) for the Minkowski functionals of general fields, and in general it is difficult to compute the value of  $c_{\text{var}}$  exactly (even if the density  $c(\ell)$  is well understood for Minkowski functionals [1]). It would be interesting to know if (2-10), or a similar relationship, holds in more generality.

**Question 2.24.** What is the relationship between  $c_{\text{var}}(\ell)$  and the derivative of the density functional  $c'_{\text{ES}}(\ell)$ ? Is  $c_{\text{var}}(\ell) \propto (\ell c'_{\text{ES}}(\ell))^2$  for the RPW?

The third question involves the asymptotic distribution of the fluctuations of  $N_{\text{ES}}(D_R, \ell)$ . For the Minkowski functionals these are known to be Gaussian in many cases (see, e.g., [14; 19; 25; 29; 34]). Non-Gaussian limit theorems have also been observed in the case of spectral singularity at the origin [27; 45].

**Question 2.25.** Does  $N_{\text{ES}}(D_R, \ell)$  have asymptotically Gaussian fluctuations? Does it depend on the field and on the level?

**2B3. Comparison to percolation models.** Recent work has established that, in many cases, the geometry of Gaussian excursion/level sets exhibits the same behaviour as the “clusters” in discrete percolation models [2; 5; 33; 44]; in particular, this is known for Gaussian fields whose correlations are positive and rapidly decaying, and has been conjectured by Bogomolny and Schmit to be true for the RPW [11]. It is therefore of interest to compare our results to what is known for percolation models.

For Bernoulli percolation on  $\mathbb{Z}^2$  with connection probability  $p \in (0, 1)$  (see [20] for background on this and other percolation models), it is known that the variance of the number of clusters in the square of side length  $R$  is of order exactly  $R^2$ . This matches the order of our lower bound on  $\text{Var}(N_{\text{ES}}(D_R, \ell))$  for Gaussian fields with positive spectral measure and rapid correlation decay, but is inconsistent with our bounds in the case of the RPW or fields with spectral singularity. In particular, our results are inconsistent with some of the stronger claims of the Bogomolny–Schmit conjecture [11; 12], which imply that the variance of  $N_{\text{ES}}(D_R, \ell)$  and  $N_{\text{LS}}(D_R, \ell)$  are of order  $R^2$  for the RPW when  $\ell$  is close to zero. On the other hand, the most important case of the Bogomolny–Schmit conjecture is the critical case, which posits that the nodal set  $\{f = 0\}$  of the RPW has statistics that match critical Bernoulli percolation ( $p = \frac{1}{2}$ ). Unfortunately our results do not cover this case.

**Question 2.26.** What is the order of  $\text{Var}(N_{\text{LS}}(D_R, 0))$  and  $\text{Var}(N_{\text{ES}}(D_R, 0))$  for the RPW? Does it agree with the Bogomolny–Schmit prediction of order  $R^2$ ?

**2C. Outline of the method.** In this section we give an outline of the proofs of our main results (Theorems 2.7 and 2.15). For clarity we focus only on the bounds for  $N_{\text{ES}}$ ; the proof for  $N_{\text{LS}}$  is near identical.

The foundation of our arguments is a versatile, elementary lemma due to Chatterjee. For random variables  $X$  and  $Y$  that take values in a common measurable space, we write  $d_{\text{TV}}(X, Y)$  to denote the total variation distance between their laws:

$$d_{\text{TV}}(X, Y) = d_{\text{TV}}(\mathbb{P}_X, \mathbb{P}_Y) = \sup\{|\mathbb{P}_X(A) - \mathbb{P}_Y(A)| : A \text{ event}\},$$

where  $\mathbb{P}_X$  and  $\mathbb{P}_Y$  denote the laws of  $X$  and  $Y$  respectively.

**Lemma 2.27** [15, Lemma 1.2]. *Let  $X$  and  $Y$  be real-valued random variables defined on the same probability space. Then, for real numbers  $a \leq b$ ,*

$$\mathbb{P}(a \leq X \leq b) \leq \frac{1}{2}(1 + \mathbb{P}(|X - Y| \leq b - a) + d_{\text{TV}}(X, Y)).$$

We combine this with the following definition:

**Definition 2.28.** Let  $X_n$  and  $Y_n$  be sequences of real-valued random variables defined on the same probability space and let  $u_n$  be a sequence of positive real numbers. We say that  $X_n$  and  $Y_n$  differ by order at least  $u_n$  if there exist constants  $c_1, c_2 > 0$  such that

$$\mathbb{P}(|X_n - Y_n| \geq c_1 u_n) \geq c_2$$

for all  $n$  sufficiently large.

**Corollary 2.29.** *Let  $X_n$  and  $Y_n$  be sequences of real-valued random variables defined on the same probability space and let  $u_n$  be a sequence of positive numbers. If  $X_n$  and  $Y_n$  differ by order at least  $u_n$  and  $d_{\text{TV}}(X_n, Y_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $X_n$  has fluctuations of order at least  $u_n$ .*

We will apply [Corollary 2.29](#) with  $X_R = N_{\text{ES}}(D_R, \ell)$  and  $Y_R = N_{\text{ES}}(D_R, \ell + a_R)$  for a certain sequence  $a_R \rightarrow 0$  as  $R \rightarrow \infty$ . There are two competing requirements on  $a_R$ : (i)  $a_R$  should decay slowly enough that  $N_{\text{ES}}(D_R, \ell)$  and  $N_{\text{ES}}(D_R, \ell + a_R)$  differ by a large order; and (ii)  $a_R$  must decay quickly enough that  $d_{\text{TV}}(N_{\text{ES}}(D_R, \ell), N_{\text{ES}}(D_R, \ell + a_R))$  tends to zero. Let us consider first the order by which  $N_{\text{ES}}(D_R, \ell)$  and  $N_{\text{ES}}(D_R, \ell + a_R)$  differ. Using the assumption that  $c_{\text{ES}}$  has nonzero (Dini) derivative at  $\ell$ , we show in [Lemma 3.4](#) that

$$|\mathbb{E}(N_{\text{ES}}(D_R, \ell) - N_{\text{ES}}(D_R, \ell + a_R))| \gtrsim R^2 a_R.$$

Using a bound on the second moment of the number of critical points in a shrinking height window from [\[32\]](#) (proven using the Kac–Rice theorem), we then show in [Lemma 3.5](#) that

$$\left(\mathbb{E}\left(\left(N_{\text{ES}}(D_R, \ell) - N_{\text{ES}}(D_R, \ell + a_R)\right)^2\right)\right)^{1/2} \lesssim R^2 a_R.$$

Since these bounds are of the same order, the second moment method implies that  $N_{\text{ES}}(D_R, \ell)$  and  $N_{\text{ES}}(D_R, \ell + a_R)$  differ by order at least  $R^2 a_R$  (see [Proposition 3.7](#)).

The next step is to bound the total variation distance between  $N_{\text{ES}}(D_R, \ell + a_R)$  and  $N_{\text{ES}}(D_R, \ell)$ . Our arguments in this step are different for general fields and for the RPW.

We shall make use of two basic properties of the total variation distance. Let  $X$  and  $Y$  be random variables taking values in a common measurable space, and let  $\mathbb{P}_X$  and  $\mathbb{P}_Y$  denote their respective laws. Then the following hold:

- Contraction: For any measurable mapping  $F$ ,

$$d_{\text{TV}}(F(X), F(Y)) \leq d_{\text{TV}}(X, Y).$$

- Pinsker’s inequality:

$$d_{\text{TV}}(X, Y) \leq \sqrt{\frac{1}{2} d_{\text{KL}}(\mathbb{P}_X \parallel \mathbb{P}_Y)}, \tag{2-11}$$

where  $d_{\text{KL}}(\mathbb{P} \parallel \mathbb{Q})$  denotes the Kullback–Leibler divergence from  $\mathbb{Q}$  to  $\mathbb{P}$ , defined as

$$d_{\text{KL}}(\mathbb{P} \parallel \mathbb{Q}) = \int \log\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right) d\mathbb{P} \tag{2-12}$$

if  $\mathbb{P}$  is absolutely continuous with respect to  $\mathbb{Q}$ , and  $d_{\text{KL}}(\mathbb{P} \parallel \mathbb{Q}) = \infty$  otherwise.

For general fields (i.e., those satisfying the conditions of [Theorem 2.7](#)), our approach is to view the random variable  $N_{\text{ES}}(D_R, \ell + a_R)$  as the number of excursion sets of the field  $f - a_R$  at level  $\ell$ , and so by the contraction property of total variation distance,

$$d_{\text{TV}}(N_{\text{ES}}(D_R, \ell), N_{\text{ES}}(D_R, \ell + a_R)) \leq d_{\text{TV}}(f, f - a_R).$$

A Cameron–Martin argument then gives an upper bound on this distance in terms of the norm of an (approximately) constant function in the reproducing kernel Hilbert space induced by the field (see [\(3-8\)](#) for the definition of this Hilbert space). By bounding this norm in terms of the behaviour of the spectral

measure at the origin, we can prove that the total variation distance  $d_{\text{TV}}(N_{\text{ES}}(D_R, \ell), N_{\text{ES}}(D_R, \ell + a_R))$  converges to zero provided that

$$a_R \ll \sqrt{g(1/R)}/R.$$

Combining this with the previous step, we deduce a fluctuation bound of order

$$R^2 a_R \approx R\sqrt{g(1/R)}.$$

In the case of the RPW (Theorem 2.15), the previous approach fails since nonzero constant functions cannot be approximated in the reproducing kernel Hilbert space of the RPW (which consists of solutions to the Helmholtz equation  $\Delta f = -f$ ). Instead, our approach is to view  $N_{\text{ES}}(D_R, \ell + a_R)$  as the number of excursion sets of the field  $\ell/(\ell + a_R)f$  at level  $\ell$  (note that this only holds for  $\ell \neq 0$ , which is why the nodal level is excluded from our results on the RPW), and so

$$d_{\text{TV}}(N_{\text{ES}}(D_R, \ell), N_{\text{ES}}(D_R, \ell + a_R)) \leq d_{\text{TV}}\left(f, \frac{\ell}{\ell + a_R} f\right). \tag{2-13}$$

Using an orthogonal expansion for the RPW in terms of Bessel functions (3-14), we show that the topological behaviour of the RPW on  $D_R$  is essentially determined by  $4R$  i.i.d. standard Gaussian variables. Pinsker’s inequality (2-11) therefore allows us to bound (2-13) in terms of the Kullback–Leibler divergence from one Gaussian vector to another. Since this divergence can be computed explicitly for Gaussian vectors, as a result we show that  $d_{\text{TV}}(N_{\text{ES}}(D_R, \ell), N_{\text{ES}}(D_R, \ell + a_R)) \rightarrow 0$  provided

$$a_R \ll 1/\sqrt{R}.$$

Combining this with the previous step, we deduce a fluctuation bound of order

$$R^2 a_R \approx R^{3/2}.$$

In both cases the main technical step is to ensure that the approximations (in the general case, approximating the constant function inside the reproducing kernel Hilbert space, and for the RPW, truncating the orthogonal expansion) do not radically change the number of excursion set components. To achieve this we apply Morse theory arguments to bound the change by the number of “quascritical points”, which we can control with local computations (see the proofs of Lemmas 3.9 and 3.11 in Section 4).

Note that the requirement that  $c_{\text{ES}}$  has nonzero (Dini) derivative is seemingly crucial to this method. In particular, it is not possible to obtain a (weaker) lower bound on the fluctuations in the case that  $c'_{\text{ES}}(\ell) = 0$ , even if we assume  $c''_{\text{ES}}(\ell) \neq 0$ , since then the second moment method fails completely (the orders of the first and second moment bounds do not match).

On the other hand, there are at least three ways in which one might try to extend our results using the described method:

- (1) First, one could prove that  $c_{\text{ES}}$  has a nonzero derivative for a larger range of levels than those in Corollaries 2.12 and 2.17.
- (2) Second, one could find other ways of bounding the total variation distance between the number of excursion sets at different levels (although we expect that our bounds are of the correct order).



- (3) Third, one could find different variables to compare in Chatterjee’s lemma. Our choice of  $Y_R = N_{\text{ES}}(D_R, \ell + a_R)$  was motivated by previous results which made an analysis of  $X_R - Y_R$  tractable, but perhaps other choices of  $Y_R$  might work.

We also believe that this method could be useful to prove fluctuations bounds on other “nonlocal” (or even “local”) geometric functionals of Gaussian fields, and in principle works equally well for Gaussian fields in higher dimensions or on manifolds.

### 3. Fluctuations of the number of excursion/level set components

In this section we prove our main results (Theorems 2.7 and 2.15) following the outline given in Section 2C, subject to two auxiliary results (Lemmas 3.9 and 3.11) whose proof is deferred to Section 4. We also give a proof of Proposition 2.19 (in Section 3C, which does not rely on the other results in Sections 3 and 4).

Recall that the lower and upper right Dini derivatives of a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  at a point  $x$  are defined respectively as

$$\partial_+ g(x) := \liminf_{\epsilon \downarrow 0} \frac{g(x + \epsilon) - g(x)}{\epsilon} \quad \text{and} \quad \partial^+ g(x) := \limsup_{\epsilon \downarrow 0} \frac{g(x + \epsilon) - g(x)}{\epsilon}. \quad (3-1)$$

The lower and upper left Dini derivatives are defined respectively as

$$\partial_- g(x) := \liminf_{\epsilon \downarrow 0} \frac{g(x) - g(x - \epsilon)}{\epsilon} \quad \text{and} \quad \partial^- g(x) := \limsup_{\epsilon \downarrow 0} \frac{g(x) - g(x - \epsilon)}{\epsilon}. \quad (3-2)$$

For the sake of simplicity, in this section we focus on  $N_{\text{ES}}$  rather than  $N_{\text{LS}}$ , and we also assume the level  $\ell$  is such that either  $\partial^+ c_{\text{ES}}(\ell) < 0$  or  $\partial_+ c_{\text{ES}}(\ell) > 0$  rather than one of the corresponding conditions for left Dini derivatives. The arguments are near identical in all of these cases, and we will mention any points of difference.

**3A. Varying the level.** We first show that  $N_{\text{ES}}(D_{R_n}, \ell)$  and  $N_{\text{ES}}(D_{R_n}, \ell + a_n)$  differ by at least a certain order, for carefully chosen sequences  $R_n \rightarrow \infty$  and  $a_n \rightarrow 0$ . There are two main inputs into this result.

The first is a deterministic topological link between  $N_{\text{ES}}(D_{R_n}, \ell)$  and  $N_{\text{ES}}(D_{R_n}, \ell + a_n)$  derived in [31]. Recall the definition of  $N_h(D_R, [a, b])$  for  $h = m^+, m^-, s^+, s^-$  given prior to Theorem 2.3. We let  $N_{\text{crit}}(D_R, [a, b])$  denote the number of critical points of  $f$  in  $D_R$  with level in  $[a, b]$ , and let  $N_{\text{tang}}(D_R)$  and  $N_{\text{tang}}(D_R, [a, b])$  denote respectively the number of critical points of  $f|_{\partial D_R}$  and those with level in  $[a, b]$ . Recall also the definition of the number of four-arm saddles  $N_{4\text{-arm}}(D_R, [a, b])$  given before Assumption 2.6.

**Lemma 3.1** [31, Corollary 2.4.7]. *Let  $f$  be a Gaussian field satisfying Assumption 2.1 and let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. Then there exist absolute constants  $c_1, c_2 > 1$  such that, for all  $R > 0$  and  $a < b$ , almost surely*

$$\begin{aligned} \left| (N_{\text{ES}}(D_R, a) - N_{\text{ES}}(D_R, b)) - (N_{m^+}(D_R, [a, b]) - N_{s^-}(D_R, [a, b])) \right| \\ \leq c_1 (N_{\text{tang}}(D_R, [a, b]) + N_{4\text{-arm}}(D_R, [a, b])) \\ \leq c_2 N_{\text{tang}}(D_R), \end{aligned} \quad (3-3)$$

so, in particular,

$$|N_{\text{ES}}(D_R, a) - N_{\text{ES}}(D_R, b)| \leq c_1(N_{\text{crit}}(D_R, [a, b]) + N_{\text{tang}}(D_R, [a, b])). \quad (3-4)$$

**Remark 3.2.** For the analogous statements for level sets (which is also given in [31, Corollary 2.4.7]), the quantity

$$N_{m^+}(D_R, [a, b]) - N_{s^-}(D_R, [a, b])$$

in (3-3) should be replaced with

$$N_{m^+}(D_R, [a, b]) - N_{s^-}(D_R, [a, b]) - N_{m^-}(D_R, [a, b]) + N_{s^+}(D_R, [a, b]).$$

We note that this lemma was technically stated in the case that  $D$  is a ball centred at the origin, however the proof in [31] also holds for rectangles. We also mention that this lemma essentially follows from the proof of [6, Lemma 2.5] (which gives the weaker inequality above and is key to proving Theorem 2.3).

The second input is a moment bound on the number of critical/tangent points of  $f$  in  $D_R$  inside shrinking height windows, which was proven in [32].

**Proposition 3.3.** *Let  $f$  be a Gaussian field satisfying Assumption 2.1 and  $D \subset \mathbb{R}^2$  a rectangle centred at the origin. Then there exists  $c > 0$  such that for all  $R > 0$  and  $a < b$ ,*

$$\begin{aligned} \mathbb{E}(N_{\text{crit}}(D_R, [a, b])^2) &< c \min\{R^4(b-a)^2 + R^2(b-a), R^4\}, \\ \mathbb{E}(N_{\text{tang}}(D_R, [a, b])^2) &< c \min\{R^2(b-a)^2 + R(b-a), R^2\}. \end{aligned}$$

*Proof.* Let  $c_D$  be the diameter of  $D$ . Then  $N_{\text{crit}}(D_R, [a, b]) \leq N_{\text{crit}}(B(c_D R), [a, b])$  and [32, Theorem 1.3] states that the second moment of the latter quantity satisfies the first inequality above.

For the second inequality, we consider the restriction of  $f$  to each of the four line segments which make up the boundary of  $D_R$ . The tangent points of  $f$  are then the critical points of the restricted field. By Cauchy–Schwarz, it is enough to prove the inequality above for each restriction separately. This is precisely the conclusion of [32, Theorem A.1].  $\square$

To apply the second moment method, we require a lower bound on the mean of the difference  $N_{\text{ES}}(D_{R_n}, \ell) - N_{\text{ES}}(D_{R_n}, \ell + a_n)$ .

**Lemma 3.4.** *Let  $f$  be a Gaussian field satisfying Assumption 2.1 and  $D \subset \mathbb{R}^2$  a rectangle centred at the origin. If  $\partial^+_{c_{\text{ES}}}(\ell) < 0$  or  $\partial^+_{c_{\text{ES}}}(\ell) > 0$  then there exists  $c > 0$  such that, for any positive sequences  $a_n \rightarrow 0$  and  $R_n \rightarrow \infty$ ,*

$$|\mathbb{E}(N_{\text{ES}}(D_{R_n}, \ell) - N_{\text{ES}}(D_{R_n}, \ell + a_n))| > cR_n^2 a_n + O(R_n)$$

for all  $n$  sufficiently large. If, in addition,  $f$  satisfies Assumption 2.6, then

$$|\mathbb{E}(N_{\text{ES}}(D_{R_n}, \ell) - N_{\text{ES}}(D_{R_n}, \ell + a_n))| > cR_n^2 a_n + O(R_n a_n) + O(\sqrt{R_n a_n})$$

for all  $n$  sufficiently large. If right Dini derivatives are replaced with left Dini derivatives, then the same conclusion holds on replacing  $N_{\text{ES}}(D_R, \ell + a_n)$  with  $N_{\text{ES}}(D_R, \ell - a_n)$ . These statements also hold if excursion sets are replaced by level sets.

*Proof.* The second inequality in (3-3) states that

$$N_{\text{ES}}(D_{R_n}, \ell) - N_{\text{ES}}(D_{R_n}, \ell + a_n) = N_{m^+}(D_{R_n}, [\ell, \ell + a_n]) - N_{s^-}(D_{R_n}, [\ell, \ell + a_n]) + O(N_{\text{tang}}(D_{R_n}))$$

almost surely, and so by [Theorem 2.3](#),

$$|\mathbb{E}(N_{\text{ES}}(D_{R_n}, \ell) - N_{\text{ES}}(D_{R_n}, \ell + a_n))| \geq |c_{\text{ES}}(\ell) - c_{\text{ES}}(\ell + a_n)| \cdot \text{Area}(D) \cdot R_n^2 + O(\mathbb{E}(N_{\text{tang}}(D_{R_n}))).$$

By [Proposition 3.3](#) and Jensen's inequality,  $\mathbb{E}(N_{\text{tang}}(D_{R_n})) \leq cR_n$  for some  $c > 0$  depending only on  $D$  and the distribution of  $f$ . Applying our assumption on the Dini derivative of  $c_{\text{ES}}$  then proves the first part of the lemma.

The tighter inequality in (3-3) states that

$$\begin{aligned} & N_{\text{ES}}(D_{R_n}, \ell) - N_{\text{ES}}(D_{R_n}, \ell + a_n) \\ &= N_{m^+}(D_{R_n}, [\ell, \ell + a_n]) - N_{s^-}(D_{R_n}, [\ell, \ell + a_n]) + O(N_{\text{tang}}(D_{R_n}, [\ell, \ell + a_n]) + N_{4\text{-arm}}(D_{R_n}, [\ell, \ell + a_n])) \end{aligned}$$

almost surely, and so by [Theorem 2.3](#) and [Assumption 2.6](#),

$$\begin{aligned} & |\mathbb{E}(N_{\text{ES}}(D_{R_n}, \ell) - N_{\text{ES}}(D_{R_n}, \ell + a_n))| \\ & \geq |c_{\text{ES}}(\ell) - c_{\text{ES}}(\ell + a_n)| \cdot \text{Area}(D) \cdot R_n^2 + o(R_n^2 a_n) + O(\mathbb{E}(N_{\text{tang}}(D_{R_n}, [\ell, \ell + a_n]))). \end{aligned}$$

By [Proposition 3.3](#) and Jensen's inequality,

$$\mathbb{E}(N_{\text{tang}}(D_{R_n}, [\ell, \ell + a_n])) = O(R_n a_n) + O(\sqrt{R_n a_n}),$$

thus proving the lemma. □

We next prove a matching second moment bound.

**Lemma 3.5.** *Let  $f$  be a Gaussian field satisfying [Assumption 2.1](#) and  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. There exists  $c_1 > 0$  such that, for any positive sequences  $a_n \rightarrow 0$  and  $R_n \rightarrow \infty$ ,*

$$\mathbb{E}((N_{\text{ES}}(D_{R_n}, \ell) - N_{\text{ES}}(D_{R_n}, \ell + a_n))^2) < c_1(R_n^4 a_n^2 + R_n^2 a_n), \quad (3-5)$$

and the same conclusion holds for level sets.

*Proof.* Combine (3-4) in [Lemma 3.1](#) with the bounds

$$\begin{aligned} \mathbb{E}(N_{\text{crit}}(D_R, [a, b])^2) &< c_1(R^4(b-a)^2 + R^2(b-a)), \\ \mathbb{E}(N_{\text{tang}}(D_R, [a, b])^2) &< c_1(R^2(b-a)^2 + R(b-a)) \end{aligned}$$

in [Proposition 3.3](#). □

**Remark 3.6.** By taking instead the bounds

$$\mathbb{E}(N_{\text{crit}}(D_R, [a, b])^2) < cR^4 \quad \text{and} \quad \mathbb{E}(N_{\text{tang}}(D_R, [a, b])^2) < cR^2$$

in [Proposition 3.3](#), and setting  $b \equiv \infty$  in (3-4), the same argument also establishes the trivial upper bound (1-1) under [Assumption 2.1](#). (To be precise, we take  $b \rightarrow \infty$  and apply the monotone convergence theorem to the squared number of critical/tangent points. We next apply the dominated convergence

theorem to  $(N_{\text{ES}}(D_R, a) - N_{\text{ES}}(D_R, b))^2$  as  $b \rightarrow \infty$ , using the squared number of critical and tangent points with height in  $[a, \infty)$  as the dominating variable.) In fact, by applying this argument to a compact domain  $B(1)$ , covering  $B(R)$  with  $\approx R^2$  copies of  $B(1)$  and controlling boundary components, we can actually derive (1-1) without the condition that  $\max_{|\alpha| \leq 2} |\partial_\alpha \kappa(x)| \rightarrow 0$  as  $|x| \rightarrow \infty$ .

Armed with matching first and second moment bounds, an application of the second moment method yields a lower bound on the order by which  $N_{\text{ES}}(D_{R_n}, \ell)$  and  $N_{\text{ES}}(D_{R_n}, \ell + a_n)$  differ.

**Proposition 3.7.** *Let  $f$  be a Gaussian field satisfying Assumption 2.1 and  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. Assume that  $\partial^+ c_{\text{ES}}(\ell) < 0$  or  $\partial_+ c_{\text{ES}}(\ell) > 0$  and let  $a_n \rightarrow 0$  and  $R_n \rightarrow \infty$  be positive sequences. Suppose that either of the following conditions hold:*

- (1)  $R_n a_n \rightarrow \infty$  as  $n \rightarrow \infty$ ;
- (2)  $f$  satisfies Assumption 2.6, and  $R_n^2 a_n$  is bounded away from zero as  $n \rightarrow \infty$ .

Then  $N_{\text{ES}}(D_{R_n}, \ell)$  and  $N_{\text{ES}}(D_{R_n}, \ell + a_n)$  differ by order at least  $R_n^2 a_n$ . This also holds if excursion sets are replaced by level sets.

*Proof.* Let

$$X_n = N_{\text{ES}}(D_{R_n}, \ell) \quad \text{and} \quad Y_n = N_{\text{ES}}(D_{R_n}, \ell + a_n).$$

Under either condition (1) or (2) in the statement of the proposition, Lemma 3.4 shows that there is a constant  $c_1 > 0$  such that

$$\mathbb{E}|X_n - Y_n| \geq |\mathbb{E}(X_n - Y_n)| > c_1 R_n^2 a_n$$

for all  $n$  sufficiently large. Combining this with the Paley–Zygmund inequality,

$$\begin{aligned} \mathbb{P}(|X_n - Y_n| > \tfrac{1}{2} c_1 R_n^2 a_n) &\geq \mathbb{P}(|X_n - Y_n| > \tfrac{1}{2} \mathbb{E}(|X_n - Y_n|)) \\ &\geq \frac{(\mathbb{E}|X_n - Y_n|)^2}{4\mathbb{E}((X_n - Y_n)^2)}. \end{aligned} \tag{3-6}$$

Combining this with Lemma 3.5 gives

$$\frac{(\mathbb{E}|X_n - Y_n|)^2}{\mathbb{E}((X_n - Y_n)^2)} \geq c_2 \frac{|\mathbb{E}(X_n - Y_n)|^2}{R_n^4 a_n^2 + R_n^2 a_n} \geq c_3 \frac{R_n^4 a_n^2}{R_n^4 a_n^2 + R_n^2 a_n} \geq c_4 > 0 \tag{3-7}$$

for constants  $c_2, c_3, c_4 > 0$  and all  $n$  sufficiently large. Combining (3-6) and (3-7) completes the proof.  $\square$

**3B. Bounding the total variation distance and completion of the proof.** We next bound the total variation distance between the number of excursion sets at different levels, using different arguments for the case of general fields (i.e., fields satisfying the conditions of Theorem 2.7) and for the RPW. This completes the proof of the main results (subject to two auxiliary lemmas, the proofs of which are deferred until Section 4).

**3B1. General fields.** We begin by recalling some general theory of Gaussian fields (for which we refer to [23]). Recall that to a continuous Gaussian field  $f$  defined on  $\mathbb{R}^d$  we can associate a Hilbert space of functions  $H \subset C(\mathbb{R}^d)$  known as the *reproducing kernel Hilbert space* (RKHS), or Cameron–Martin space, defined as the completion of the space of finite linear combinations of the covariance function  $\kappa$

$$\sum_{1 \leq i \leq n} a_i \kappa(s_i, \cdot), \quad a_i \in \mathbb{R}, \quad s_i \in \mathbb{R}^d, \tag{3-8}$$

equipped with the inner product

$$\left\langle \sum_{1 \leq i \leq n} a_i K(s_i, \cdot), \sum_{1 \leq j \leq m} a'_j K(s'_j, \cdot) \right\rangle_H = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} a_i a'_j K(s_i, s'_j).$$

The importance of the RKHS for our purposes is this corollary of the Cameron–Martin theorem:

**Proposition 3.8.** *Let  $f$  be a continuous Gaussian field defined on some Euclidean space. For every  $h \in H$ ,*

$$d_{\text{TV}}(f, f + h) \leq \frac{1}{2} \|h\|_H.$$

*Proof.* Let  $\mathbb{P}$  and  $\mathbb{Q}$  denote respectively the laws of  $f$  and  $f + h$ . The Cameron–Martin theorem (see [23, Theorems 14.1, 3.33]) states that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\cdot) = \exp\{I(h)(\cdot) - \frac{1}{2} \|h\|_H^2\},$$

where  $I(h)$  is a certain random variable (known as the Paley–Weiner integral) whose law under  $\mathbb{P}$  is a centred Gaussian variable with variance  $\|h\|_H^2$ . Hence

$$d_{\text{KL}}(\mathbb{P} \parallel \mathbb{Q}) = \int \log \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{P} = - \int I(h) d\mathbb{P} + \frac{1}{2} \|h\|_H^2 = \frac{1}{2} \|h\|_H^2,$$

and the result follows from Pinsker’s inequality (2-11). □

If the Gaussian field  $f$  is stationary, the norm  $\|h\|_H$  can be written explicitly in terms of the spectral measure  $\mu$ . Indeed we can represent  $H$  as the Fourier transform of  $L^2_{\text{sym}}(d\mu)$ , the space of complex Hermitian functions square integrable with respect to  $\mu$ . Specifically, each  $h \in H$  is of the form  $\mathcal{F}(\hat{h} d\mu)$  with a unique  $\hat{h} \in L^2_{\text{sym}}(d\mu)$ , and

$$\langle h_1, h_2 \rangle_H = \langle \hat{h}_1, \hat{h}_2 \rangle_{L^2_{\text{sym}}(d\mu)}.$$

In particular,

$$\|h\|_H = \|\hat{h}\|_{L^2_{\text{sym}}(d\mu)}.$$

As an immediate consequence, in the case that  $\mu$  has density  $\rho$ , we have that  $\mathcal{F}(\hat{h} d\mu) = \mathcal{F}(\hat{h} \rho dx)$ ; i.e.,  $\hat{h}$  differs from the standard (inverse) Fourier transform  $\mathcal{F}^{-1}(h)$  by division by  $\rho$ . If  $\Omega := \text{supp}(\hat{h})$  has finite area, this implies the bound

$$\begin{aligned} \|h\|_H^2 &= \|\hat{h}\|_{L^2_{\text{sym}}(d\mu)}^2 \leq \sup\{|\mathcal{F}^{-1}(h)(x)|^2 / \rho(x) : x \in \Omega\} \text{Area}(\Omega) \\ &\leq \frac{\sup\{|\mathcal{F}^{-1}(h)(x)|^2 : x \in \Omega\} \text{Area}(\Omega)}{\inf\{\rho(x) : x \in \Omega\}}. \end{aligned} \tag{3-9}$$

We now restrict to the setting of [Theorem 2.7](#), letting  $f$  be a Gaussian field satisfying [Assumptions 2.1](#) and [2.5](#). We recall that  $g(r) := \inf_{x \in B(2r)} \rho(x)$ . For each  $r > 0$ , we define  $h_r : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$h_r(t) = \frac{1}{4r^2} \mathcal{F}[\mathbb{1}_{[-r,r]^2}](t) = \frac{\sin(2\pi r t_1)}{2\pi r t_1} \frac{\sin(2\pi r t_2)}{2\pi r t_2}. \quad (3-10)$$

Since  $\mu$  has a density  $\rho$  which is uniformly positive near the origin, we see that for  $r > 0$  sufficiently small,  $h_r$  is an element of the RKHS  $H$ . Then, by [\(3-9\)](#),

$$\|h_r\|_H \leq \frac{1}{2r\sqrt{g(r)}}. \quad (3-11)$$

We will use  $h_r$ , as  $r \rightarrow 0$ , to approximate the constant function 1; if we choose positive sequences  $R_n \rightarrow \infty$  and  $r_n \rightarrow 0$  such that  $r_n R_n \rightarrow 0$ , then by a Taylor expansion,

$$\|1 - h_{r_n}\|_{C^2(D_{R_n})} = O(r_n^2 R_n^2) \quad \text{as } n \rightarrow \infty. \quad (3-12)$$

In the next lemma we show that this approximation has a negligible effect on the number of excursion sets; i.e., the number of excursion sets of  $f - a_n$  is well approximated by the number of excursion sets of  $f - a_n h_{r_n}$  for an appropriate choice of  $r_n$ . We extend our previous notation  $N_{\text{ES}}(D_R, \ell)$  slightly, defining  $N_{\text{ES}}(g; D_R, \ell)$  for  $g \in C_{\text{loc}}^2(\mathbb{R}^2)$  to be the number of components of  $\{g \geq \ell\}$  contained in  $D_R$  (so that  $N_{\text{ES}}(D_R, \ell) = N_{\text{ES}}(f; D_R, \ell)$ ).

**Lemma 3.9.** *Let  $f$  be a Gaussian field satisfying the conditions of [Theorem 2.7](#) and let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin. Fix  $\ell \in \mathbb{R}$  and let  $R_n, r_n$  and  $a_n$  be sequences of positive numbers such that  $R_n \rightarrow \infty$ ,  $r_n \rightarrow 0$ ,  $a_n \rightarrow 0$  and  $r_n R_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then there exist  $c, n_0 > 0$  such that, for all  $n > n_0$ ,*

$$\mathbb{E}(|N_{\text{ES}}(f - a_n; R_n, \ell) - N_{\text{ES}}(f - a_n h_{r_n}; R_n, \ell)|) < c a_n r_n^2 R_n^4.$$

*The same conclusion holds for level sets.*

We defer the proof of [Lemma 3.9](#) until [Section 4](#). The upshot is that the conclusion of [Proposition 3.7](#) also holds when we replace  $f - a_n$  with  $f - a_n h_{r_n}$ .

Our final ingredient for proving [Theorem 2.7](#) is the following elementary lemma:

**Lemma 3.10.** *Let  $X_n$  be a sequence of random variables and  $u_n$  a sequence of positive real numbers. If  $X_n$  has fluctuations of order at least  $\delta_n u_n$  for all positive sequences  $\delta_n$  converging to zero arbitrarily slowly, then  $X_n$  has fluctuation of order at least  $u_n$ .*

*Proof.* We note that by [Definition 2.4](#),  $X_n$  has fluctuations of order  $u_n$  if and only if  $X_n/u_n$  has fluctuations of order 1. Therefore we may assume  $u_n = 1$  for all  $n$ .

We fix some positive sequence  $\delta_k \rightarrow 0$  as  $k \rightarrow \infty$ , and suppose that  $X_n$  does not have fluctuations of order at least 1. Then by [Definition 2.4](#), for each  $k$  we can find  $n_k > n_{k-1}$  and  $a_{n_k} < b_{n_k}$  such that  $b_{n_k} - a_{n_k} \leq \delta_k^2$  and

$$\mathbb{P}(a_{n_k} \leq X_{n_k} \leq b_{n_k}) > 1 - \delta_k^2. \quad (3-13)$$

By assumption,  $X_{n_k}$  has fluctuations of order  $\delta_k$ . So there exist absolute constants  $c_1, c_2 > 0$  such that for all  $k$  sufficiently large (so that  $\delta_k < c_1$ ) we have  $b_{n_k} - a_{n_k} \leq c_1 \delta_k$  and hence

$$\mathbb{P}(a_{n_k} \leq X_{n_k} \leq b_{n_k}) \leq \delta_k.$$

Provided  $k$  is large enough, this contradicts (3-13), so we deduce that  $X_n$  has fluctuations of order 1, as required.  $\square$

*Proof of Theorem 2.7.* Let  $R_n \rightarrow \infty$  and  $\delta_n \rightarrow 0$  be positive, monotone sequences such that  $\delta_n^2 R_n \rightarrow \infty$ . If  $g(r) \rightarrow \infty$  as  $r \rightarrow 0$  we also choose  $\delta_n$  converging to zero sufficiently slowly that  $\delta_n^2 \sqrt{g(\delta_n/R_n)} \rightarrow \infty$  as  $n \rightarrow \infty$ . We apply Proposition 3.7 with  $a_n = \delta_n^2 \sqrt{g(\delta_n/R_n)}/R_n$  and deduce that  $N_{\text{ES}}(f; D_{R_n}, \ell)$  and  $N_{\text{ES}}(f; D_{R_n}, \ell + a_n)$  differ by order at least  $R_n^2 a_n = \delta_n^2 R_n \sqrt{g(\delta_n/R_n)}$ . Applying Lemma 3.9 with  $r_n = \delta_n/R_n$  and Markov's inequality we have that, for every  $\epsilon > 0$  and  $n$  sufficiently large,

$$\mathbb{P}(|N_{\text{ES}}(f - a_n; D_{R_n}, \ell) - N_{\text{ES}}(f - a_n h_{r_n}; D_{R_n}, \ell)| > \epsilon R_n^2 a_n) < \frac{c a_n r_n^2 R_n^4}{\epsilon R_n^2 a_n} = c \delta_n^2 / \epsilon \rightarrow 0.$$

Hence  $N_{\text{ES}}(f; D_{R_n}, \ell)$  and  $N_{\text{ES}}(f - a_n h_{r_n}; D_{R_n}, \ell)$  also differ by order at least  $R_n^2 a_n$ . Moreover, by Proposition 3.8 and (3-11), there is a  $c_0 > 0$  such that

$$\begin{aligned} d_{\text{TV}}(N_{\text{ES}}(f; D_{R_n}, \ell), N_{\text{ES}}(f - a_n h_{r_n}; D_{R_n}, \ell)) &\leq d_{\text{TV}}(f, f - a_n h_{r_n}) \\ &\leq c_0 \|a_n h_{r_n}\|_H \\ &\leq c_0 a_n / (2r_n \sqrt{g(r_n)}) = (c_0/2) \delta_n \rightarrow 0, \end{aligned}$$

where we implicitly restrict all fields to the domain  $D_{R_n}$  when calculating the total variation distance. Therefore, by Corollary 2.29, we conclude that  $N_{\text{ES}}(D_{R_n}, \ell)$  has fluctuations of order at least  $R_n^2 a_n$ . Since  $\delta_n$  can be chosen to converge to zero arbitrarily slowly, Lemma 3.10 implies that  $N_{\text{ES}}(D_{R_n}, \ell)$  has fluctuations of order at least  $R_n \sqrt{g(1/R_n)}$ , completing the proof of the theorem.  $\square$

**3B2. The random plane wave.** We now move onto the proof of Theorem 2.15. It is known that the RPW has the orthogonal expansion

$$f(x) = \sum_{m \in \mathbb{Z}} a_m J_{|m|}(r) e^{im\theta}, \quad (3-14)$$

where  $(r, \theta)$  represents  $x$  in polar coordinates,  $J_m$  is the  $m$ -th Bessel function and  $a_m = b_m + i c_m = \overline{a_{-m}}$  with  $b_0, (\sqrt{2}b_m)_{m \in \mathbb{N}}$  and  $(\sqrt{2}c_m)_{m \in \mathbb{N}}$  independent standard (real) Gaussians and  $c_0 = 0$ . (This function is clearly Gaussian and can be shown to have the correct covariance structure using Graf's addition theorem for Bessel functions.) Since Bessel functions decay exponentially quickly in  $m$  (see (3-16) below) the series converges locally uniformly with probability one.

We will use a truncation of this expansion to approximately parametrise  $f$  using a finite number of random variables, but we do so in a slightly unusual way: letting  $d_k$  be a sequence of independent standard Gaussian variables, we then write  $a_0 = b_0 = \sum_{k=1}^{\infty} 2^{-k/2} d_k$  and for  $N \in \mathbb{N}$  define

$$f_N(x) = \sum_{1 \leq k \leq N} 2^{-k/2} d_k J_0(r) + \sum_{1 \leq |m| \leq N} a_m J_{|m|}(r) e^{im\theta}. \quad (3-15)$$

The first summation here will clearly approximate  $a_0 J_0(r)$  with exponentially small error as  $N \rightarrow \infty$ . Known inequalities for Bessel functions [47, Section 8.5, (9)] state that, for all  $\alpha \in (0, 1)$ ,  $m \geq 0$ , and  $r < \alpha m$ ,

$$|J_m(r)| \leq c_1 e^{-c_2 m}, \quad (3-16)$$

so that the terms beyond  $m \approx 2R$  are exponentially small inside  $B(R)$ . So overall we see that  $f_N$  will give a very accurate approximation to  $f$  on appropriate domains. The reason for approximating  $a_0$  incrementally (as opposed to just truncating (3-14) after  $N$  terms) is technical; it ensures that the remainder  $f - f_N$  is nondegenerate at the origin (note that  $J_n(0) = 0$  for all  $n \geq 1$ ), which will simplify some of our arguments.

In the next lemma we show that these terms have a bounded effect on the number of excursion sets.

**Lemma 3.11.** *Let  $f$  be the random plane wave, and let  $D \subset \mathbb{R}^2$  be an open rectangle centred at the origin with diameter  $\text{Diam}(D)$ . Fix  $\ell^* \in \mathbb{R}$  and  $\beta \in (0, 1)$ . Then there exist  $c, n_0 > 0$  such that, for all  $\ell \in [\ell^* - 1, \ell^* + 1]$ ,  $N \geq n_0$  and  $R \cdot \text{Diam}(D) \leq \beta N$ ,*

$$\mathbb{E}(|N_{\text{ES}}(f; D_R, \ell) - N_{\text{ES}}(f_N; D_R, \ell)|) < c.$$

The proof of Lemma 3.11 is deferred to Section 4. Armed with this lemma we can complete the proof of Theorem 2.15.

*Proof of Theorem 2.15.* Let  $R_n \rightarrow \infty$  be a positive sequence, and  $a_n = \delta_n R_n^{-1/2}$  for some sequence  $\delta_n > 0$  which converges to zero slowly enough that  $R_n a_n \rightarrow \infty$  (we will eventually allow  $\delta_n$  to converge to zero arbitrarily slowly). Applying Proposition 3.7 shows that  $N_{\text{ES}}(f; D_{R_n}, \ell)$  and  $N_{\text{ES}}(f; D_{R_n}, \ell + a_n)$  differ by order at least  $R_n^2 a_n = \delta_n R_n^{3/2}$ .

Now choose  $m_n = \lceil 2 \text{Diam}(D) \cdot R_n \rceil$  (where  $\lceil x \rceil$  denotes the least integer greater than or equal to  $x$ ). Applying Lemma 3.11 and Markov's inequality shows that for any  $\epsilon > 0$

$$\mathbb{P}(|N_{\text{ES}}(f; D_{R_n}, \ell) - N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell)| > \epsilon R_n^2 a_n) \rightarrow 0$$

and

$$\mathbb{P}(|N_{\text{ES}}(f; D_{R_n}, \ell + a_n) - N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell + a_n)| > \epsilon R_n^2 a_n) \rightarrow 0$$

as  $n \rightarrow \infty$ , so we conclude that  $N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell)$  and  $N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell + a_n)$  also differ by order at least  $\delta_n R_n^{3/2}$ .

For  $n$  sufficiently large (so that  $\ell + a_n$  is bounded away from zero),  $f_{m_n}(t) \geq \ell + a_n$  if and only if  $\ell/(\ell + a_n) f_{m_n}(t) \geq \ell$ . Therefore

$$N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell + a_n) = N_{\text{ES}}\left(\frac{\ell}{\ell + a_n} f_{m_n}; D_{R_n}, \ell\right).$$

Since  $f_{m_n}$  is parametrised by  $3m_n$  independent standard Gaussian variables we see that

$$\begin{aligned} d_{\text{TV}}(N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell), N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell + a_n)) &= d_{\text{TV}}\left(N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell), N_{\text{ES}}\left(\frac{\ell}{\ell + a_n} f_{m_n}; D_{R_n}, \ell\right)\right) \\ &\leq d_{\text{TV}}\left(\mathcal{N}(0, I_{3m_n}), \mathcal{N}\left(0, \left(\frac{\ell}{\ell + a_n}\right)^2 I_{3m_n}\right)\right), \end{aligned}$$



where in the final expression,  $d_{\text{TV}}$  denotes the total variation distance between  $3m_n$ -dimensional random vectors. By Pinsker's inequality (2-11), the square of the above quantity is at most

$$\frac{1}{2}d_{\text{KL}}\left(\mathcal{N}\left(0, \left(\frac{\ell}{\ell+a_n}\right)^2 I_{3m_n}\right) \parallel \mathcal{N}(0, I_{3m_n})\right), \quad (3-17)$$

where  $d_{\text{KL}}$  denotes the Kullback–Leibler divergence defined by (2-12). If  $\mathbb{P}$  and  $\mathbb{Q}$  are two centred  $k$ -dimensional nondegenerate Gaussian measures with covariances  $\Sigma_1$  and  $\Sigma_2$  correspondingly then, letting  $X \sim \mathbb{P}$ ,

$$\begin{aligned} d_{\text{KL}}(\mathbb{P} \parallel \mathbb{Q}) &= \int \log\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right) d\mathbb{P} \\ &= \mathbb{E}\left(-\frac{1}{2} \log\left(\frac{\det \Sigma_1}{\det \Sigma_2}\right) - \frac{1}{2} X^t \Sigma_1^{-1} X + \frac{1}{2} X^t \Sigma_2^{-1} X\right) \\ &= \frac{1}{2} \left(\log\left(\frac{\det \Sigma_2}{\det \Sigma_1}\right) - \mathbb{E}(\text{Tr}(\Sigma_1^{-1} X X^t) - \text{Tr}(\Sigma_2^{-1} X X^t))\right) \\ &= \frac{1}{2} \left(\log\left(\frac{\det \Sigma_2}{\det \Sigma_1}\right) - k + \text{Tr}(\Sigma_2^{-1} \Sigma_1)\right), \end{aligned}$$

where in the second line we used that a scalar is equal to its trace, and the trace operator  $\text{Tr}(\cdot)$  is invariant under cyclic permutations, and in the third line we used that  $\mathbb{E}(\text{Tr}(\Sigma_i^{-1} X X^t)) = \text{Tr}(\Sigma_i^{-1} \mathbb{E}(X X^t)) = \text{Tr}(\Sigma_i^{-1} \Sigma_i)$  by linearity. Therefore the quantity in (3-17) is equal to

$$\frac{3m_n}{4} \left( \left(\frac{\ell}{\ell+a_n}\right)^2 - 1 - \ln\left(\left(\frac{\ell}{\ell+a_n}\right)^2\right) \right) \leq \frac{c_1}{4} (6R_n + 1) a_n^2$$

for some constant  $c_1 > 0$  depending only on  $\ell$  (the inequality follows from a Taylor expansion of the logarithm). This bound converges to zero as  $n \rightarrow \infty$  and so we can apply Corollary 2.29 and conclude that  $N_{\text{ES}}(f_{m_n}; D_{R_n}, \ell)$  has fluctuations of order at least  $\delta_n R_n^{3/2}$ . Applying Lemma 3.11, the same conclusion is true for  $N_{\text{ES}}(f; D_{R_n}, \ell)$ . Since  $\delta_n$  can be chosen to converge to zero arbitrarily slowly, Lemma 3.10 implies that  $N_{\text{ES}}(f; D_{R_n}, \ell)$  has fluctuations of order at least  $R_n^{3/2}$ , as required.  $\square$

**3C. Proof of Proposition 2.19.** We now give a proof of Proposition 2.19 that essentially follows from the law of large numbers for excursion set components (Theorem 2.2). We note that it is also possible to prove this result using the methods from Section 3.

*Proof of Proposition 2.19.* Let  $f$  be a Gaussian field satisfying Assumption 2.18, which means it can be represented as  $f = g + \sqrt{\alpha}Z$  where  $g$  has spectral measure  $\nu^*$  and  $Z$  is an independent standard Gaussian variable. We note that by Theorem 2.3,  $c_{\text{ES}}(\nu^*, \ell)$  is continuous in  $\ell$  and tends to zero as  $\ell \rightarrow \infty$ . Furthermore, by [6, Corollary 1.12],  $c_{\text{ES}}(\nu^*, \ell)$  is positive for some  $\ell \in \mathbb{R}$  (actually for any  $\ell > 0$ ). Combining these facts we see that there exists  $\epsilon > 0$  and open sets  $U_1, U_2 \subset \mathbb{R}$  such that for all  $\ell_1 \in U_1$  and  $\ell_2 \in U_2$

$$c_{\text{ES}}(\nu^*, \ell_1) > 4\epsilon \quad \text{and} \quad c_{\text{ES}}(\nu^*, \ell_2) < \epsilon.$$

We now choose an arbitrary  $\ell \in \mathbb{R}$  and a positive, increasing sequence  $R_n \rightarrow \infty$ . Using the independence of  $Z$  and  $g$ , and denoting the standard Gaussian probability density by  $\phi$ , we see that

$$\begin{aligned} \mathbb{P}(N_{\text{ES}}(f; D_{R_n}, \ell) \geq 3\epsilon \text{Area}(D)R_n^2) &= \mathbb{P}(N_{\text{ES}}(g; D_{R_n}, \ell - \sqrt{\alpha}Z) \geq 3\epsilon \text{Area}(D)R_n^2) \\ &= \int_{\mathbb{R}} \mathbb{P}(N_{\text{ES}}(g; D_{R_n}, \ell - \sqrt{\alpha}x) \geq 3\epsilon \text{Area}(D)R_n^2) \phi(x) dx \\ &= \int_{\mathbb{R}} \mathbb{P}(N_{\text{ES}}(g; D_{R_n}, u) \geq 3\epsilon \text{Area}(D)R_n^2) \phi\left(\frac{\ell-u}{\sqrt{\alpha}}\right) \frac{1}{\sqrt{\alpha}} du \\ &\geq \int_{U_1} \mathbb{P}(N_{\text{ES}}(g; D_{R_n}, u) \geq 3\epsilon \text{Area}(D)R_n^2) \phi\left(\frac{\ell-u}{\sqrt{\alpha}}\right) \frac{1}{\sqrt{\alpha}} du. \end{aligned}$$

Since  $g$  satisfies [Assumption 2.1](#), [Theorem 2.2](#) implies that for each  $u \in U$ , the probability in the integrand above converges to one as  $n \rightarrow \infty$ . Applying Fatou’s lemma (and the fact that  $U$  is open, so has positive Lebesgue measure) we see that the integral above is bounded below by a positive constant for all sufficiently large  $n$ .

Applying an identical argument to  $U_2$  shows that  $\mathbb{P}(N_{\text{ES}}(f; D_{R_n}, \ell) \leq 2\epsilon \text{Area}(D)R_n^2)$  is bounded below by a positive constant for all sufficiently large  $n$ . These two bounds show that  $N_{\text{ES}}(f; D_{R_n}, \ell)$  has fluctuations of order at least  $R_n^2$  (by definition). Since  $\ell \in \mathbb{R}$  and  $R_n \uparrow \infty$  were arbitrary, combining this with the trivial upper bound on the variance (1-1) (see also [Remark 3.6](#)) completes the proof of the result. Identical arguments apply to level sets. □

### 4. Perturbation arguments

In this section we prove [Lemmas 3.9](#) and [3.11](#), thus completing the proof of all results in the paper.

We begin with some heuristics. Let  $F$  be a realisation of a random field on a compact domain  $D$ . Our aim is to control the expected difference between the number of components of  $\{F \geq 0\}$  and  $\{F - p \geq 0\}$ , where  $p$  is a small (possibly random) perturbation. If  $p$  is a constant function taking the value  $c > 0$ , then the standard methods of Morse theory show that the difference between these two quantities is at most the number of critical points of  $F$  with level between 0 and  $c$  (because the excursion set  $\{F \geq \ell\}$  varies continuously with  $\ell$  unless passing through a critical point of  $F$ , in which case the number of components changes by at most one). Since the number of critical points is a local quantity, we can use the Kac–Rice formula to bound its mean.

This same reasoning can be applied to more general perturbations  $p$ . Assuming some regularity of  $F$  and  $p$  (which will be specified below) the number of components of  $\{F - \alpha p \geq 0\}$  changes continuously with  $\alpha$  unless passing through a value at which 0 is a critical level of  $F - \alpha p$ , and it can be shown that at such points the number of components changes by at most one. Therefore the difference in the number of excursion sets is bounded above by the number of points at which  $F - \alpha p = 0$  and  $\nabla(F - \alpha p) = 0$  for some  $\alpha \in [0, 1]$  (plus an analogous term which controls boundary effects, as we work with a bounded domain). Under our nondegeneracy assumptions, with probability one there is at most one such point for each  $\alpha$ , and for all but finitely many values of  $\alpha$  there are no such points. Since the number of these points is still a local quantity, an application of the Kac–Rice formula will yield [Lemmas 3.9](#) and [3.11](#).

Let us formalise the concepts just described. Let  $\mathcal{D} \subset \mathbb{R}^2$  be an open rectangle, and let  $F$  and  $p$  be  $C^2$  functions defined on a neighbourhood of  $\overline{\mathcal{D}}$ . Let  $\mathcal{C}$  denote the set of corners of  $\overline{\mathcal{D}}$  and  $\mathcal{E} := \partial\mathcal{D} \setminus \mathcal{C}$  denote the edges of  $\overline{\mathcal{D}}$ . We refer to  $\mathcal{D}$ ,  $\mathcal{E}$  and  $\mathcal{C}$  as the strata of  $\overline{\mathcal{D}}$ . We say that  $x$  in a neighbourhood of  $\overline{\mathcal{D}}$  is a quasicritical point of  $(F, p)$  at level  $\alpha \in [0, 1]$  if  $(F - \alpha p)(x) = 0$  and  $x$  is a critical point of  $F - \alpha p$  restricted to the stratum of  $\overline{\mathcal{D}}$  (or  $\mathbb{R}^2 \setminus \overline{\mathcal{D}}$ ) containing  $x$ . So for  $x \in \mathcal{D} \cup (\mathbb{R}^2 \setminus \overline{\mathcal{D}})$ , this condition says that  $(F - \alpha p)(x) = 0$  and  $\nabla(F - \alpha p)(x) = 0$ . For  $x \in \mathcal{E}$  this means that  $(F - \alpha p)(x) = 0$  and  $\nabla(F - \alpha p)(x)$  is normal to  $\partial\mathcal{D}$  at  $x$ . For  $x \in \mathcal{C}$  this just says that  $(F - \alpha p)(x) = 0$ ; the other condition holds vacuously. We say that a quasicritical point  $x$  at level  $\alpha$  is nondegenerate if  $x$  is a nondegenerate critical point of  $F - \alpha p$  restricted to the stratum containing  $x$  (so, for example, if  $x \in \mathcal{D}$ , this just says that  $\det \nabla^2(F - \alpha p)(x) \neq 0$ ).

We let  $N_{\text{QC}}(F; p, \overline{\mathcal{D}})$  denote the number of quasicritical points of  $(F, p)$  in  $\overline{\mathcal{D}}$  (at any level  $\alpha \in [0, 1]$ ). If we replace  $\overline{\mathcal{D}}$  with a stratum ( $\mathcal{D}$ ,  $\mathcal{E}$  or  $\mathcal{C}$ ) in this expression, then we mean the number of quasicritical points restricted to that stratum. For  $x \in \mathcal{E}$ , let  $v_{\partial}(x)$  and  $v_{\bar{\partial}}(x)$  denote respectively the unit vectors in the tangent and normal directions to  $\partial\mathcal{D}$ , and let  $\nabla_{\partial}$  and  $\nabla_{\bar{\partial}}$  denote the derivatives in these respective directions.

**Assumption 4.1.** Let  $\mathcal{D} \subset \mathbb{R}^2$  be an open rectangle. Let  $F$  and  $p$  be  $C^2$  functions defined on a neighbourhood of  $\overline{\mathcal{D}}$  which satisfy the following:

- (1) The quasicritical points of  $(F, p)$  in a neighbourhood of  $\overline{\mathcal{D}}$  are nondegenerate, all occur at distinct levels and are not contained in  $\{p = 0\}$ .
- (2) If  $x \in \partial\mathcal{D}$  is a quasicritical point of  $(F, p)$  at level  $\alpha$  then  $\nabla(F - \alpha p)(x) \neq 0$ ; if, in addition,  $x \in \mathcal{C}$  then  $\nabla(F - \alpha p)(x)$  is not parallel to either edge of  $\mathcal{D}$  which joins at  $x$ .

**Theorem 4.2.** Let  $\mathcal{D} \subset \mathbb{R}^2$  be an open rectangle and let  $(F, p)$  satisfy [Assumption 4.1](#). Then

$$|N_{\text{ES}}(F; \overline{\mathcal{D}}, 0) - N_{\text{ES}}(F - p; \overline{\mathcal{D}}, 0)| \leq N_{\text{QC}}(F, p, \overline{\mathcal{D}}).$$

The same conclusion holds on replacing  $N_{\text{ES}}$  with  $N_{\text{LS}}$ .

This theorem is only a slight generalisation of known results from Morse theory and so its proof is given in the [Appendix](#).

We can now state our main perturbation result.

**Proposition 4.3.** Let  $\mathcal{D} \subset \mathbb{R}^2$  be an open rectangle and  $\mathcal{D}_+$  be a fixed compact neighbourhood of  $\overline{\mathcal{D}}$ . Suppose that  $F$  and  $p$  are independent  $C^3$ -smooth planar Gaussian fields defined on  $\mathcal{D}_+$  satisfying the following conditions:

- (1) For each  $x, y \in \mathcal{D}_+$  with  $x \neq y$ , the Gaussian vector  $(F(x), F(y), \nabla F(x), \nabla F(y))$  is nondegenerate.
- (2) For each  $x \in \mathcal{D}_+$ , the Gaussian vector  $(\nabla F(x), \nabla^2 F(x))$  is nondegenerate.
- (3) For each  $x \in \mathcal{E}$ , the vectors  $(F(x), \nabla_{\partial} F(x), \nabla_{\partial} \nabla_{\bar{\partial}} F(x))$  and  $(F(x), \nabla_{\bar{\partial}} F(x), \nabla_{\bar{\partial}}^2 F(x))$  are nondegenerate (as Gaussians).
- (4) Either
  - (a)  $p$  is deterministic and the set  $\{p = 0\} \cap \mathcal{D}_+$  consists of a finite union of isolated points, or
  - (b) for each  $x \in \mathcal{D}_+$ , the Gaussian vector  $(p(x), \nabla p(x))$  is nondegenerate.

Then with probability one

$$|N_{\text{ES}}(F; \mathcal{D}, 0) - N_{\text{ES}}(F - p; \mathcal{D}, 0)| \leq N_{\text{QC}}(F; p, \bar{\mathcal{D}}).$$

The same conclusion holds on replacing  $N_{\text{ES}}$  with  $N_{\text{LS}}$ .

Given [Theorem 4.2](#), the proof of this result is a straightforward application of Bulinskaya's lemma to various combinations of  $F$ ,  $p$  and their first two derivatives.

*Proof.* It is sufficient to verify that  $F$  and  $p$  almost surely satisfy both parts of [Assumption 4.1](#), since then [Theorem 4.2](#) yields the result.

(1) We first verify that, almost surely, the quasicritical points of  $(F, p)$  are at distinct levels. We define  $\mathcal{D}_n = \{(x, y) \in \mathcal{D}_+^2 : |x - y| \geq 1/n\} \times [0, 1]$  and  $g_1 : \mathcal{D}_n \rightarrow \mathbb{R}^6$  by

$$g_1(x, y, \alpha) = \begin{pmatrix} \nabla(F - \alpha p)(x) \\ \nabla(F - \alpha p)(y) \\ (F - \alpha p)(x) \\ (F - \alpha p)(y) \end{pmatrix}.$$

Bulinskaya's lemma [[1](#), Lemma 11.2.10] states that  $g_1$  almost surely does not hit  $0 \in \mathbb{R}^6$  at any point in  $\mathcal{D}_n$  provided it is almost surely  $C^1$  and that the density of  $g_1(x, y, \alpha)$  is bounded on a neighbourhood of 0 uniformly in  $\mathcal{D}_n$ . We know that  $g_1 \in C^1(\mathcal{D}_n)$  by assumption and so turn to the second condition. Since  $F$  and  $p$  are independent, the density of  $g_1(x, y, \alpha)$  can be given by a convolution over the densities of

$$(\nabla F(x), \nabla F(y), F(x), F(y)) \quad \text{and} \quad \alpha(\nabla p(x), \nabla p(y), p(x), p(y)), \quad (4-1)$$

and therefore it is sufficient to show that the density of either of these vectors is bounded. By assumption, the covariance matrix of the first vector in (4-1) is nondegenerate for every  $x \neq y$ , and by continuity the determinant of this covariance matrix is bounded away from zero on the compact set  $\mathcal{D}_n$ . Since this vector is Gaussian, this implies that its density is uniformly bounded above and allows us to apply Bulinskaya's lemma. Taking a countable union over  $n$  completes the proof for quasicritical points in  $\mathcal{D} \cup (\mathcal{D}_+ \setminus \bar{\mathcal{D}})$ .

Applying the same argument to  $g_2 : \mathcal{E}^2 \times [0, 1] \rightarrow \mathbb{R}^4$  and  $g_3 : \mathcal{E} \times \mathcal{D}_+ \times [0, 1] \rightarrow \mathbb{R}^5$  defined by

$$g_2 := \begin{pmatrix} \nabla_{\partial}(F - \alpha p)(x) \\ \nabla_{\partial}(F - \alpha p)(y) \\ (F - \alpha p)(x) \\ (F - \alpha p)(y) \end{pmatrix} \quad \text{and} \quad g_3 := \begin{pmatrix} \nabla_{\partial}(F - \alpha p)(x) \\ \nabla(F - \alpha p)(y) \\ (F - \alpha p)(x) \\ (F - \alpha p)(y) \end{pmatrix},$$

proves that, almost surely, the quasicritical points of  $(F, p)$  in  $\mathcal{D} \cup (\mathcal{D}_+ \setminus \bar{\mathcal{D}})$  and  $\mathcal{E}$  are all at distinct levels. Considering similar functions shows that the quasicritical points in the corners  $\mathcal{C}$  also occur at disjoint levels, but we omit this for brevity.

Applying the arguments above to  $g_4 : \mathcal{E} \times [0, 1] \rightarrow \mathbb{R}^3$  given by

$$g_4 := \begin{pmatrix} \nabla_{\partial}^2(F - \alpha p)(x) \\ \nabla_{\partial}(F - \alpha p)(x) \\ (F - \alpha p)(x) \end{pmatrix}$$

shows that the quasicritical points of  $(F, p)$  in  $\mathcal{E}$  are nondegenerate almost surely. A slightly different version of Bulinskaya’s lemma [3, Proposition 6.5] states that since  $g_5 : \mathcal{D}_+ \times [0, 1] \rightarrow \mathbb{R}^3$  defined by

$$g_5 := \begin{pmatrix} \nabla(F - \alpha p)(x) \\ (F - \alpha p)(x) \end{pmatrix}$$

is almost surely  $C^2$  and has a uniformly bounded univariate probability density, there is almost surely no  $(x, \alpha) \in \mathcal{D}_+ \times [0, 1]$  such that  $g_5(x, \alpha) = 0$  and  $\det \nabla^2(F - \alpha p)(x) = 0$ . Hence the quasicritical points of  $(F, p)$  in  $\mathcal{D} \cup (\mathcal{D}_+ \setminus \overline{\mathcal{D}})$  are nondegenerate almost surely. Finally, any quasicritical points in  $\mathcal{C}$  are vacuously nondegenerate.

It remains to show that  $(F, p)$  has no quasicritical points in  $\{p = 0\}$ . In the case that  $p$  is deterministic, we apply Bulinskaya’s lemma to  $g_5$  restricted to  $(\mathcal{D}_+ \cap \{p = 0\}) \times [0, 1]$  and  $g_6 : (\partial \mathcal{D} \cap \{p = 0\}) \times [0, 1] \rightarrow \mathbb{R}^2$  defined by

$$g_6 := \begin{pmatrix} \nabla_{\partial}(F - \alpha p)(x) \\ (F - \alpha p)(x) \end{pmatrix},$$

which gives the result. (Technically in the first case we need only consider one component of  $\nabla(F - \alpha p)$  since our domain is one-dimensional.) In the case that the variance of  $p(x)$  is nonzero for each  $x$  (and so bounded away from zero by compactness), we define  $g_7 : \mathcal{D}_+ \times [0, 1] \rightarrow \mathbb{R}^4$  and  $g_8 : \mathcal{E} \times [0, 1] \rightarrow \mathbb{R}^3$  by

$$g_7 := \begin{pmatrix} \nabla(F - \alpha p)(x) \\ F(x) \\ p(x) \end{pmatrix} \quad \text{and} \quad g_8 := \begin{pmatrix} \nabla_{\partial}(F - \alpha p)(x) \\ F(x) \\ p(x) \end{pmatrix}.$$

Once again, Bulinskaya’s lemma (along with the convolution argument) gives the result. For the four points in  $\mathcal{C}$  we note that  $p$  is nonzero almost surely.

(2) Applying the same arguments to  $g_5$  restricted to  $\mathcal{E} \times [0, 1]$  shows that  $(F, p)$  almost surely has no quasicritical points in  $\mathcal{E}$  such that  $\nabla(F - \alpha p) = 0$ . Similarly restricting  $g_6$  to  $\mathcal{C} \times [0, 1]$  (where we define  $\nabla_{\partial}$  in the appropriate way for each point) proves the second part of this condition.  $\square$

We now prove Lemmas 3.9 and 3.11. We state these proofs exclusively for excursion sets; the proofs for level sets are identical. The simpler case is Lemma 3.9, since the perturbation is deterministic.

*Proof of Lemma 3.9 (given Proposition 4.3).* For fixed  $n$  sufficiently large, we define

$$F = f - \ell - a_n \quad \text{and} \quad p = a_n(h_{r_n} - 1).$$

Then by the definition of  $h_{r_n}$  in (3-10), the zero set of  $p|_{D_{R_n}}$  consists of a single point at the origin (this also requires  $n$  to be sufficiently large). Hence we may apply Proposition 4.3 to the functions  $F$  and  $p$  on the domain  $\mathcal{D} = D_{R_n}$  ( $f$  satisfies the conditions of the proposition by Assumptions 2.1 and 2.5), and so it is sufficient to prove that there exists a  $c > 0$  such that, for  $n$  sufficiently large,

$$\mathbb{E}(N_{\text{QC}}(F; D_{R_n}, p)) < ca_n r_n^2 R_n^4 \quad \text{and} \quad \mathbb{E}(N_{\text{QC}}(F; \partial D_{R_n}, p)) < ca_n r_n^2 R_n^3. \tag{4-2}$$

We begin with the first bound in (4-2). Define  $G : D_{R_n} \times [0, 1] \rightarrow \mathbb{R}^3$  by

$$G(x, \alpha) = \begin{pmatrix} \nabla(F - \alpha p)(x) \\ (F - \alpha p)(x) \end{pmatrix}, \quad (4-3)$$

and let  $p_{G(x, \alpha)}$  denote the density of the (nondegenerate) Gaussian vector  $G(x, \alpha)$ .

We now apply the Kac–Rice formula to  $G$ . Specifically we apply [3, Theorem 6.2 and Proposition 6.5] (which require  $G$  to be  $C^2$  almost surely) to conclude that

$$\begin{aligned} & \mathbb{E}(N_{\text{QC}}(F; D_{R_n}, p)) \\ &= \iint_{D_{R_n} \times [0, 1]} \mathbb{E} \left( \left| \det \begin{pmatrix} \nabla^2(F - \alpha p)(x) & \nabla(F - \alpha p)(x) \\ -\nabla p(x)^t & -p(x) \end{pmatrix} \right| \middle| G(x, \alpha) = 0 \right) p_{G(x, \alpha)}(0) \, d\alpha \, dx \\ &\leq \text{Area}(D) R_n^2 \sup_{\substack{x \in D_{R_n} \\ \alpha \in [0, 1]}} \mathbb{E} \left( \left| \det \begin{pmatrix} \nabla^2(F - \alpha p)(x) & \nabla(F - \alpha p)(x) \\ -\nabla p(x)^t & -p(x) \end{pmatrix} \right| \middle| G(x, \alpha) = 0 \right) p_{G(x, \alpha)}(0). \end{aligned}$$

The density  $p_{G(x, \alpha)}(0)$  is bounded above by  $c_1/\sqrt{\det \Sigma(x, \alpha)}$ , where  $\Sigma(x, \alpha)$  is the covariance matrix of  $G(x, \alpha)$  and  $c_1 > 0$  is an absolute constant. Using the condition  $G(x, \alpha) = 0$  (which implies that  $\nabla(F - \alpha p)(x) = 0$ ), we have

$$\begin{aligned} & \mathbb{E} \left( \left| \det \begin{pmatrix} \nabla^2(F - \alpha p)(x) & \nabla(F - \alpha p)(x) \\ -\nabla p(x)^t & -p(x) \end{pmatrix} \right| \middle| G(x, \alpha) = 0 \right) \\ &= |p(x)| \cdot \mathbb{E} \left( \left| \det(\nabla^2(F - \alpha p)(x)) \right| \middle| G(x, \alpha) = 0 \right) \\ &\leq c_2 |p(x)| \max_{|k|=2} \max \{ \mathbb{E}(|\partial^k F(x)|^2 \mid G(x, \alpha) = 0), |\partial^k p(x)|^2 \}, \end{aligned}$$

where  $c_2 > 0$  is an absolute constant and we have expanded the determinant and applied Hölder’s inequality in the final line. By (3-12),

$$\|p\|_{C^2(D_{R_n})} = O(a_n(r_n R_n)^2) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence it suffices to show that

$$\frac{1}{\det(\Sigma(x, \alpha))} \quad \text{and} \quad \max_{|k|=2} \mathbb{E}(|\partial^k F(x)|^2 \mid G(x, \alpha) = 0) \quad (4-4)$$

are bounded above, uniformly over  $n$  sufficiently large and  $(x, \alpha) \in D_{R_n} \times [0, 1]$ .

Considering the first term, since  $p$  is deterministic and  $f$  is stationary,

$$\Sigma(x, \alpha) = \text{Cov} \begin{pmatrix} \nabla f(x) \\ f(x) \end{pmatrix} = \text{Cov} \begin{pmatrix} \nabla f(0) \\ f(0) \end{pmatrix},$$

which has a nonzero determinant, by Assumption 2.1, that does not depend on  $x$  or  $\alpha$ .

Now turning to the second term of (4-4), since  $f$  is stationary,  $(f(x), \nabla^2 f(x))$  is independent of  $\nabla f(x)$  (this is a standard result for stationary Gaussian fields; see [1, Chapter 5]) and so for  $|k| = 2$

$$\begin{aligned} \mathbb{E}(|\partial^k F(x)|^2 \mid G(x, \alpha) = 0) &= \mathbb{E}(|\partial^k f(x)|^2 \mid f(x) = \ell + a_n + \alpha p(x)) \\ &= \mathbb{E}(|\partial^k f(0)|^2 \mid f(0) = \ell + a_n + \alpha p(x)), \end{aligned} \quad (4-5)$$

where the second equality also follows from stationarity of  $f$  (recall that  $p$  is deterministic). By Gaussian regression [3, Proposition 1.2], this final expression is continuous in the variable  $\ell + a_n + \alpha p(x)$ . Since  $a_n$  and  $\|p\|_{C^2(D_{R_n})}$  converge to zero as  $n \rightarrow \infty$ , we see that this variable has a uniformly bounded range for all  $n$  and all  $(x, \alpha) \in D_{R_n} \times [0, 1]$ . Therefore (4-5) is uniformly bounded as required. This establishes the first bound in (4-2).

We prove the second bound in (4-2) for the number of quasicritical points in each of the four edges (i.e., line segments) that make up  $\partial D_{R_n}$  separately. Let  $F_\partial$  and  $p_\partial$  be the restrictions of  $F$  and  $p$  respectively to one of these line segments (viewed as one-dimensional fields). We apply the Kac–Rice formula to

$$G_\partial(x, \alpha) = \begin{pmatrix} (F'_\partial - \alpha p'_\partial)(x) \\ (F_\partial - \alpha p_\partial)(x) \end{pmatrix},$$

which gives the result once we establish that

$$\mathbb{E}(|F''_\partial(x)| \mid G_\partial(x, \alpha) = 0), \quad |p''_\partial(x)| \quad \text{and} \quad \frac{1}{\det(\Sigma_\partial(x, \alpha))}$$

are bounded above uniformly over  $n$  sufficiently large and  $(x, \alpha) \in [0, cR_n] \times [0, 1]$ , where  $\Sigma_\partial(x, \alpha)$  denotes the covariance matrix of  $G_\partial(x, \alpha)$  and we have parametrised the line segment by  $[0, cR_n]$  for some  $c$  depending only on  $D$ . These facts are proven using arguments near identical to those given above. □

The proof of Lemma 3.11 is slightly more complex because the perturbation is random, and we require an additional lemma to control its behaviour:

**Lemma 4.4.** *Let  $f$  be the random plane wave and  $f_N$  the approximation of this expansion given by (3-15). For each  $\beta \in (0, 1)$  and  $k \in \mathbb{N}$  there exists  $c_1, c_2 > 0$  such that, for all  $N \geq 1$ ,*

$$\mathbb{E}(\|f - f_N\|_{C^k(B(\beta N))}^2) \leq c_1 e^{-c_2 N}.$$

*Proof.* Recall from (3-16) that there exist  $c_1, c_2 > 0$  such that, for all  $m \geq 0$  and  $r \leq \alpha m$ ,

$$|J_m(r)| \leq c_1 e^{-c_2 m}. \tag{4-6}$$

Applying this to the orthogonal expansion (3-14), along with the fact that  $|J_0| \leq 1$  we have

$$\begin{aligned} \mathbb{E}\left(\sup_{x \in B(\beta N)} |f(x) - f_N(x)|^2\right) &\leq \mathbb{E}\left(\left(\sum_{j>N} 2^{-j/2} |d_j| + \sum_{|m|>N} c_1 |a_m| e^{-c_2 |m|}\right)^2\right) \\ &\leq c_3 e^{-c_4 N} \end{aligned}$$

for some  $c_3, c_4 > 0$ , which gives the result in the case  $k = 0$ .

For the general case  $k \geq 1$ , we differentiate in polar coordinates and use Bessel identities to replace the resulting terms by linear combinations of Bessel functions. For instance, since

$$\frac{2m J_m(r)}{r} = J_{m-1}(r) + J_{m+1}(r) \quad \text{and} \quad 2J'_m(r) = J_{m-1}(r) - J_{m+1}(r),$$

we have that

$$\begin{aligned} & \partial_{x_1} f(x) - \partial_{x_1} f_N(x) \\ &= \sum_{j>N} -2^{-j/2} d_j J_1(r) \cos(\theta) + \sum_{|m|>N} a_m e^{im\theta} (J'_{|m|}(r) \cos(\theta) - i \sin(\theta)(m/r) J_{|m|}(r)) \\ &= \sum_{j>N} -2^{-j/2} d_j J_1(r) \cos(\theta) + \sum_{|m|>N} \frac{1}{2} a_m e^{im\theta} ((J_{|m|-1}(r) - J_{|m|+1}(r)) \cos(\theta) \\ & \qquad \qquad \qquad - i \sin(\theta) \operatorname{sgn}(m)(J_{|m|-1}(r) + J_{|m|+1}(r))). \end{aligned}$$

Hence by the triangle inequality and (4-6), we have

$$\mathbb{E} \left( \sup_{x \in B(\beta N)} |\partial_{x_1} f(x) - \partial_{x_1} f_N(x)|^2 \right) \leq c_5 e^{-c_6 N}.$$

The proof for  $k > 1$  is similar, and we omit the details. □

*Proof of Lemma 3.11 (assuming Proposition 4.3).* We recall the orthogonal expansion of the random plane wave in (3-14) and its finite approximation  $f_N$  in (3-15). We define

$$F := f_N - \ell \quad \text{and} \quad p := f_N - f,$$

which are independent  $C^\infty$  Gaussian fields. We wish to apply Proposition 4.3 with  $\mathcal{D} = D_R$ . We first observe that for any  $x \in \mathcal{D}_+$ , one can easily check that  $(p(x), \nabla p(x))$  is nondegenerate using the orthogonal expansion in (3-14) (note that by isotropy, it is enough to check nondegeneracy when  $x = (x_1, 0)$ , which easily follows from independence of the variables  $b_n$  and  $c_n$ ). This means that the fourth condition of Proposition 4.3 holds.

The first three conditions of Proposition 4.3 hold if we replace  $F$  with  $f - \ell$  (the first two conditions follow from Assumption 2.1 and by stationarity the third condition is verified by computing the partial derivatives of order at most four of  $\kappa$  at the origin). By Lemma 4.4, the covariance function of  $F = f_N - \ell$  and its first two derivatives converge uniformly on  $B(\beta N)$  to those of  $f - \ell$ . This implies that once  $N$  is sufficiently large,  $F$  satisfies the same nondegeneracy conditions as  $f - \ell$  on all  $D_R$  such that  $R \cdot \operatorname{Diam}(D) \leq \beta N$ .

We have therefore verified all the conditions of Proposition 4.3 and conclude that with probability one,

$$|N_{\text{ES}}(F; D_R, 0) - N_{\text{ES}}(F - p; D_R, 0)| \leq N_{\text{QC}}(F; p, \overline{D_R}).$$

To complete the proof of the lemma, it suffices to show that

$$\mathbb{E}(N_{\text{QC}}(F; p, D_R)) < c_1 \quad \text{and} \quad \mathbb{E}(N_{\text{QC}}(F; p, \partial D_R)) < c_1 \tag{4-7}$$

uniformly over  $N$  sufficiently large and  $R \cdot \operatorname{Diam}(D) \leq \beta N$ . Arguing as in the proof of Lemma 3.9 (with one additional application of Hölder's inequality because  $p$  is now random), the first bound in (4-7) follows from the Kac–Rice formula once we check the following two conditions. First, for all  $N$  sufficiently large

$$\sup_{x \in B(\beta N)} \mathbb{E}(|p(x)|^2 \mid G(x, \alpha) = 0)^{1/2} \leq \frac{1}{N^2}, \tag{4-8}$$



where  $G(x, \alpha)$  is defined as in (4-3) with  $p_{G(x,\alpha)}$  its density. Second, we require that for all multi-indices  $k$  with  $|k| = 2$

$$\mathbb{E}(|\partial^k F(x)|^4 | G(x, \alpha) = 0) p_{G(x,\alpha)}(0) \quad \text{and} \quad \mathbb{E}(|\partial^k p(x)|^4 | G(x, \alpha) = 0) p_{G(x,\alpha)}(0) \quad (4-9)$$

are bounded above uniformly over  $(x, \alpha) \in B(\beta N) \times [0, 1]$  for sufficiently large  $N$ .

Since  $(p, G)$  is jointly Gaussian, the distribution of  $p$  conditional on  $G$  is also Gaussian. Therefore in order to prove (4-8), it is enough to show that

$$\sup_{x \in B(\beta N)} \max \{ \text{Var}(p(x) | G(x, \alpha) = 0), \mathbb{E}(p(x) | G(x, \alpha) = 0)^2 \} \leq \frac{1}{N^4}.$$

Conditioning on some elements of a Gaussian vector can only reduce the variance of other elements, so

$$\text{Var}(p(x) | G(x, \alpha) = 0) \leq \text{Var}(p(x)),$$

and the supremum of this quantity over  $x \in B(\beta N)$  decays exponentially in  $N$  by Lemma 4.4. Turning to the expectation, by Gaussian regression

$$\mathbb{E}(p(x) | G(x, \alpha) = 0) = \text{Cov} \left( p(x), \begin{pmatrix} \nabla(f_N - \alpha p)(x) \\ (f_N - \alpha p)(x) \end{pmatrix} \right) \text{Var} \begin{pmatrix} \nabla(f_N - \alpha p)(x) \\ (f_N - \alpha p)(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \ell \end{pmatrix}.$$

By Lemma 4.4, as  $N \rightarrow \infty$  the covariance of  $(\nabla(f_N - \alpha p)(x), (f_N - \alpha p)(x))$  converges uniformly to that of  $(\nabla f(x), f(x))$  (which is constant, since  $f$  is stationary). Therefore by the Cauchy–Schwarz inequality, the above expression is bounded above, for all  $N$  sufficiently large, by  $c\mathbb{E}(p(x)^2)^{1/2}$  for some constant  $c$  depending only on  $\ell$ . Applying Lemma 4.4 then completes the proof of (4-8).

A near identical argument shows that the conditional expectations in each term of (4-9) are uniformly bounded for  $N$  sufficiently large. It therefore remains to show that  $p_{G(x,\alpha)}(0)$  is uniformly bounded. Since this is a Gaussian density, we need only show that the determinant of  $\text{Var}(G(x, \alpha))$  is uniformly bounded away from zero. As  $N \rightarrow \infty$ , this covariance matrix converges uniformly to  $\text{Var}(\nabla f(x), f(x))$ , which has a nonzero determinant that does not depend on  $x$ . This implies the necessary bound and so completes the proof of the first bound in (4-7).

For the second bound in (4-7), we count the quasicritical points on each component of  $\partial D_R$  separately and bound their expectation using the same argument as above applied to the boundary.  $\square$

### Appendix: Morse theory arguments

In this appendix we prove Theorem 4.2. Throughout this appendix we always assume that  $F$ ,  $p$  and  $\mathcal{D}$  satisfy Assumption 4.1. The results that we need are very similar to classical results from Morse theory. The main difference is that we work with functions on compact domains with boundary and instead of considering one function at different levels we consider a smooth family of functions at a single level. The first modification is addressed by stratified Morse theory. The second is less studied, but very closely related results have appeared before. The statement that we need is not genuinely new, but we were unable to find a reference which is applicable in exactly our case. The most relevant results are in [21; 46].

We start with a very natural statement: as we vary  $\alpha$ , the set where  $g_\alpha := F - \alpha p \geq 0$  changes continuously (in particular, its topology does not change), unless  $\alpha$  passes through a value such that there is a quasicritical point at this level. Our proof of this statement is very similar to the standard flow arguments used in Morse theory.

Let us define  $A_\alpha = \{x \in \overline{\mathcal{D}} : g_\alpha(x) \geq 0\}$ . Our first claim is the following:

**Proposition A.1.** *Let  $A_\alpha$  be as above and suppose that  $(F, p)$  has no quasicritical points at level  $\alpha$  for  $\alpha \in [\alpha_1, \alpha_2]$ . Then  $A_{\alpha_1}$  and  $A_{\alpha_2}$  have the same number of connected components not intersecting  $\partial\mathcal{D}$ .*

*Proof.* We will prove that the number of connected components of  $A_\alpha$  which do not intersect  $\partial\mathcal{D}$  is continuous in  $\alpha$ , and hence constant. Let us define the flow

$$\frac{dx_\alpha}{d\alpha} = -\partial_\alpha g_\alpha(x_\alpha) \frac{\nabla g_\alpha(x_\alpha)}{|\nabla g_\alpha(x_\alpha)|^2}. \tag{A-1}$$

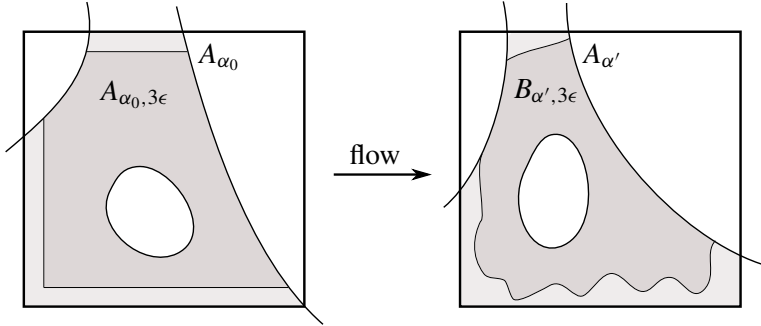
This defines a smooth flow away from the critical points of  $g_\alpha$ . An application of the chain rule shows that  $g_\alpha(x_\alpha)$  is constant in  $\alpha$ .

By  $\mathcal{D}_\epsilon$  we denote the set of points of  $\mathcal{D}$  that are a distance at least  $\epsilon$  away from  $\partial\mathcal{D}$  and we let  $A_{\alpha,\epsilon} = A_\alpha \cap \mathcal{D}_\epsilon$ . By ‘‘quasicritical point of  $g_\alpha$ ’’ we just mean a quasicritical point of  $(F, p)$  at level  $\alpha$  (recall that quasicritical points are defined prior to [Assumption 4.1](#)). We emphasise that the quasicritical points of  $g_\alpha$  include but do not coincide with the critical points of  $g_\alpha$ , with the latter being defined in the usual way (as points  $y$  satisfying  $\nabla g_\alpha(y) = 0$ ).

Since  $g_\alpha$  has no quasicritical points we see that all critical points of  $g_\alpha$  are separated from its nodal set (the set where  $g_\alpha = 0$ ). Furthermore these critical points and the nodal set depend continuously on  $\alpha$  (as we describe and justify below). We therefore claim that for every  $\alpha_0 \in (\alpha_1, \alpha_2)$ , we can find  $\epsilon, \delta > 0$  such that the following hold for all  $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$ :

- (1) All components of  $\{g_\alpha = 0\} \cap \overline{\mathcal{D}}$  are either contained in  $\mathcal{D}_{6\epsilon}$ , or intersect  $\partial\mathcal{D}_{\epsilon'}$  transversally for all  $0 \leq \epsilon' \leq 6\epsilon$  (if the component intersects a corner, this means that the line is transversal to both boundary intervals). All components of  $\{g_\alpha = 0\} \cap (\overline{\mathcal{D}} \setminus \mathcal{D}_{6\epsilon})$  have exactly one point on  $\partial\mathcal{D}$ .
- (2) All critical points of  $g_\alpha$  are at least distance  $2\epsilon$  away from  $\{g_\alpha = 0\} \cap \mathcal{D}$  (this also includes critical points outside of  $\mathcal{D}$ ). There are no critical points of  $g_\alpha$  in  $\mathcal{D}_\epsilon \setminus \mathcal{D}_{6\epsilon}$ .
- (3) If we start flow [\(A-1\)](#) from a point  $x_{\alpha_0}$  which is in an  $\epsilon$ -neighbourhood of the boundary of  $A_{\alpha_0, 3\epsilon}$  then  $|x_{\alpha_0} - x_\alpha| < \epsilon$ .

**Proof of (1):** By compactness, there is a small neighbourhood of the boundary such that all components of the restriction of the nodal set  $\{g_{\alpha_0} = 0\}$  to this neighbourhood also intersect  $\partial\mathcal{D}$ . Since there are no quasicritical points of  $g_{\alpha_0}$  on the boundary, all nodal lines (i.e., curves in the nodal set  $\{g_{\alpha_0} = 0\}$ ) that intersect  $\partial\mathcal{D}$  do so transversally. By transversality, the gradient of  $g_{\alpha_0}$  at the point where the nodal line intersects the boundary is not orthogonal to the boundary. By continuity, the same is true in some neighbourhood. This means that the nodal line intersects  $\partial\mathcal{D}_{\epsilon'}$  transversally for all sufficiently small  $\epsilon'$ . Since ‘‘sufficiently small’’ here depends on the second derivatives of  $g_{\alpha_0}$ , by uniform continuity, we can choose  $\epsilon$  such that the above claim holds simultaneously for all  $\alpha$  sufficiently close to  $\alpha_0$  and all  $\epsilon' \leq 6\epsilon$ .



**Figure 1.** The gradient flow (A-1) continuously deforms  $\partial A_{\alpha_0, 3\epsilon}$  to  $\partial B_{\alpha', 3\epsilon}$ . Clearly  $B_{\alpha', 3\epsilon} \subset A_{\alpha'}$ . Note that the flow might be undefined *inside*  $A_{\alpha', 3\epsilon}$ : we only move the boundary (although the flow is really defined and controlled everywhere except small neighbourhoods of critical points, so we can move not only the boundary, but also most of the domain by this flow, and it is easy to extend the flow in a continuous way to these small neighbourhoods). Light grey parts are quadrilaterals that can be contracted to the corresponding arcs on the boundary of  $A_{\alpha_0, 3\epsilon}$  and  $B_{\alpha', 3\epsilon}$  respectively.

Proof of (2): It is a standard result that nondegenerate critical points of a smooth family of functions depend continuously on the parameter. Hence, for all  $\alpha$  sufficiently close to  $\alpha_0$ , all critical points of  $g_\alpha$  are  $\epsilon$ -close to the critical points of  $g_{\alpha_0}$ . Since there are no critical points of  $g_{\alpha_0}$  on the nodal lines, there are no critical points in some neighbourhood of  $\{g_\alpha = 0\} \cap \mathcal{D}$ . If  $g_{\alpha_0}$  has no critical points on  $\partial\mathcal{D}$ , then by the same argument  $g_\alpha$  has no critical points in some neighbourhood of  $\partial\mathcal{D}$  for all  $\alpha$  sufficiently close to  $\alpha_0$ . Finally, if  $g_{\alpha_0}$  has any critical points on the boundary, then by the same argument as above, for all  $\alpha$  sufficiently close to  $\alpha_0$  all critical points of  $g_\alpha$  are either in a very small neighbourhood of the boundary or outside of a larger neighbourhood of the boundary.

Proof of (3): The last part follows from the second one. By compactness,  $\partial_\alpha g_\alpha$  is uniformly bounded. Since  $x_{\alpha_0}$  is away from critical points, we have a uniform lower bound on  $|\nabla g_\alpha|$ , and hence the speed of the flow is uniformly bounded. This means that if  $\delta$  is sufficiently small, then the flow cannot move by more than  $\epsilon$ . Note that the flow is not limited to  $\mathcal{D}$ , it is defined globally outside of critical points. We allow the flow to start in  $\mathcal{D}$  and leave it.

From now on we assume that  $\alpha_0$ ,  $\epsilon$  and  $\delta$  are chosen so that these three statements hold. Since  $g_\alpha$  has no critical points at level zero, the sets  $\{g_\alpha(x) = 0\}$  are made up of  $C^2$ -smooth curves that either intersect the boundary of  $\mathcal{D}$  transversely or do not intersect it at all. This means that  $A_\alpha \setminus A_{\alpha, 3\epsilon}$  is a disjoint union of quadrilaterals (by “quadrilateral” we mean a simply connected domain bounded by a simple piecewise smooth curve with four marked points; these points are “vertices” and the arcs between them are “edges” of the quadrilateral). For each component of  $A_\alpha \cap \partial\mathcal{D}$  there is a quadrilateral such that its four “sides” are this boundary component, two subarcs of  $\{g_\alpha = 0\}$  and a part of  $\partial\mathcal{D}_{3\epsilon}$  (see Figure 1). Since each quadrilateral can be retracted to its side,  $A_{\alpha, \epsilon}$  is a deformation retract of  $A_\alpha$ .

Let us consider  $\alpha' \in [\alpha_0 - \delta, \alpha_0 + \delta]$ . As  $\alpha_0$  changes to  $\alpha'$ , the flow continuously moves the boundary of  $A_{\alpha_0, 3\epsilon}$  by at most  $\epsilon$ . This gives a homotopy of  $\partial A_{\alpha_0, 3\epsilon}$  and its image. Since the flow is planar, the image

of  $\partial A_{\alpha_0, 3\epsilon}$  is the boundary of a domain which is homotopic to  $A_{\alpha_0, 3\epsilon}$ . We denote this domain by  $B_{\alpha', 3\epsilon} \subset \mathcal{D}$ . Since the flow is continuous and preserves  $g_\alpha(x_\alpha)$  the following holds: the parts of  $\partial A_{\alpha_0, 3\epsilon}$  which are made of nodal lines of  $g_{\alpha_0}$  move to nodal lines of  $g_{\alpha'}$  and the straight parts of  $\partial A_{\alpha_0, 3\epsilon}$ , where  $g_{\alpha_0} \geq 0$ , become smooth curves inside  $A_{\alpha'}$ . Since the flow does not move by more than  $\epsilon$ , these curves are actually inside  $\mathcal{D}_{2\epsilon}$ . Moreover, since the flow is invertible,  $A_{\alpha', 4\epsilon} \subset B_{\alpha', 3\epsilon} \subset A_{\alpha', 2\epsilon}$  and  $B_{\alpha', 3\epsilon} \cap \mathcal{D}_{4\epsilon}$  is equal to  $A_{\alpha', 4\epsilon}$ . In the same way as before,  $A_{\alpha'} \setminus B_{\alpha', 3\epsilon}$  is made of quadrilaterals that can be retracted. This proves that  $B_{\alpha', 3\epsilon}$  is a deformation retract of  $A_{\alpha'}$ , so they have the same topology. This proves that  $A_{\alpha_0}$  and  $A_{\alpha'}$  have the same topology.

Finally we note that in the above argument, the components of  $A_{\alpha_0}$  which do not intersect  $\partial\mathcal{D}$  are mapped by the flow to components of  $A_{\alpha'}$  which do not intersect  $\partial\mathcal{D}$ . Therefore the number of such components is nondecreasing as we move from  $\alpha_0$  to  $\alpha'$ . By considering the inverse of flow (A-1), we see that the same is true when moving from  $\alpha'$  to  $\alpha_0$ . This argument proves that the number of connected components which intersect  $\partial\mathcal{D}$  is continuous and hence constant.  $\square$

Next we would like to analyse what happens when  $\alpha$  passes through a critical level (i.e., the level of a quasicritical point). Again, this is very similar to standard Morse theory arguments which give a full CW complex decomposition of excursion sets. Since we are working in the two-dimensional case and only care about the number of connected components, many arguments can be significantly simplified.

We will control the effects of quasicritical points locally by using a version of the Morse lemma for manifolds with corners. First we require some definitions from [21]. Recall that for  $z \in K$  a submanifold with corners,  $j(z)$  is the number such that a neighbourhood of  $z$  can be mapped by a chart to a neighbourhood of zero in  $[0, \infty)^{j(z)} \times \mathbb{R}^{n-j(z)}$ . For each  $j$ , we define the set of  $z \in K$  with  $j(z) = j$  to be a stratum of  $K$ . So in the case that  $K = \overline{\mathcal{D}}$ , the three strata are the interior  $\mathcal{D}$ , the edges  $\mathcal{E}$  and the corners  $\mathcal{C}$  (agreeing with our earlier definition of strata for rectangles).

**Definition A.2.** Let  $M$  be a smooth  $n$ -dimensional manifold,  $K$  be a submanifold with corners and  $g : K \rightarrow \mathbb{R}$  be a  $C^2$  function. We say that  $g$  is a Morse function if it satisfies the following:

- (1) If  $H$  is a stratum of  $K$  then any critical points of  $g|_H$  are nondegenerate.
- (2) If  $H_1$  and  $H_2$  are (distinct) strata of  $K$  with  $z \in H_1 \subset \overline{H_2}$  then  $z$  is not a critical point of  $g|_{\overline{H_2}}$ .

**Lemma A.3** (Morse lemma [21, Lemma 5]). *Let  $M$  be a smooth  $n$ -dimensional manifold,  $K$  be a submanifold with corners and  $g : K \rightarrow \mathbb{R}$  be a Morse function. If  $z \in K$  is a critical point of  $g$  restricted to the stratum of  $K$  containing  $z$  then there exists  $\lambda \in \{0, 1, \dots, n - j(z)\}$  and a chart  $(U, (x_1, \dots, x_n))$  with  $z \in U$  such that*

$$g = g(z) - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_{n-j(z)}^2 + (-1)^{\sigma(n-j(z)+1)} x_{n-j(z)+1} + \dots + (-1)^{\sigma(n)} x_n$$

*holds in  $U \cap K$  for some  $\sigma(n - j(z) + 1), \dots, \sigma(n) \in \mathbb{N}$ . We define this number  $\lambda$  to be the index of the critical point  $z$ .*

We note that [21, Lemma 5] is stated slightly differently (with a fixed sign in front of the terms  $x_{n-j(z)+1}, \dots, x_n$ ) because it is stated under a certain assumption on the signs of components of  $\nabla g$ .

We also comment that we only use this lemma in the case  $n = 2$ , however for notational convenience we will split the local variables  $x_i$  into three groups as above. The convention, of course, will be that some of these  $x_i$  are identically zero.

If  $F$  and  $p$  satisfy [Assumption 4.1](#), then for any  $x \in \overline{\mathcal{D}}$  such that  $p(x) \neq 0$ , there is a neighbourhood of  $x$  on which  $F/p$  is the restriction of a Morse function on  $\overline{\mathcal{D}}$ . This follows easily from considering the gradient and Hessian of  $F/p$ .

**Proposition A.4.** *Let  $\mathcal{D} \subset \mathbb{R}^2$  be an open rectangle and let  $(F, p)$  satisfy [Assumption 4.1](#). Suppose that  $(F, p)$  has a quasicritical point at level  $\beta$ . Then, for  $\delta > 0$  sufficiently small, the number of connected components of  $A_{\beta+\delta}$  which do not intersect  $\partial\mathcal{D}$  differs from the corresponding number for  $A_{\beta-\delta}$  by at most one (and the corresponding number for  $A_\beta$  is between these two). The same conclusion holds for the level sets  $\{F - \alpha p = 0\} \cap \overline{\mathcal{D}}$  for  $\alpha \in \{\beta - \delta, \beta + \delta, \beta\}$ .*

This result is almost a corollary of [\[21, Theorem 8\]](#), which gives a CW-decomposition of excursion sets, but our setting is slightly different, so their result cannot be applied as stated in our situation. It is possible to modify the proof of [\[21, Theorem 8\]](#) in order to prove [Proposition A.4](#). We do not follow this road. Instead we give a different proof based on [Lemma A.3](#) and the flow argument used in [Proposition A.1](#).

*Proof.* By [Proposition A.1](#), it is enough to prove the result for some  $\delta > 0$  such that there are no quasicritical points with level in  $[\beta - \delta, \beta + \delta] \setminus \{\beta\}$ . Let  $z$  denote the quasicritical point of  $(F, p)$  at level  $\beta$  and choose  $\delta > 0$  such that this is the only quasicritical point with level in  $[\beta - \delta, \beta + \delta]$  (this is possible because  $\overline{\mathcal{D}}$  is compact and quasicritical points are isolated). Since  $p(z) \neq 0$  by [Assumption 4.1](#), we can find a neighbourhood  $U$  of  $z$  on which  $p \neq 0$ , and hence  $F/p$  is well defined on this neighbourhood. We will assume  $p(z) > 0$ ; if this value was negative our arguments could be repeated with some sign changes. Since

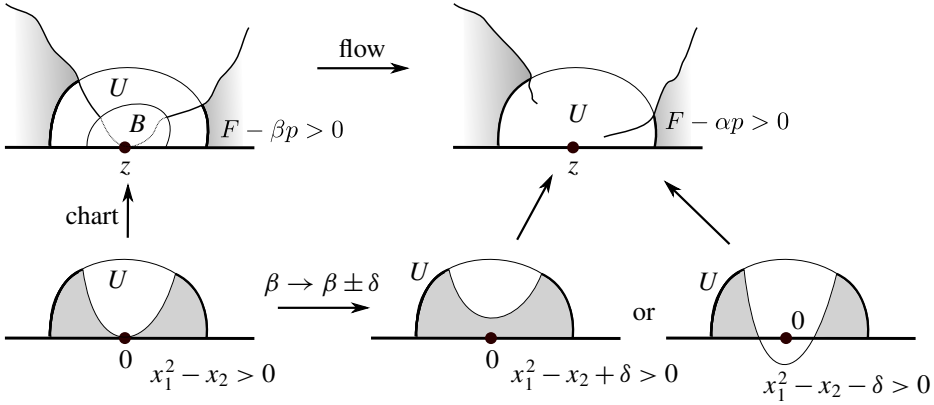
$$\nabla(F/p) = \frac{1}{p^2}(p\nabla F - F\nabla p) = \frac{1}{p}(\nabla F - (F/p)\nabla p),$$

it is clear that  $F - \alpha p$  has a quasicritical point at  $x$  where  $p(x) \neq 0$  if and only if  $x$  is a critical point of  $F/p$  restricted to the stratum of  $x$  at level  $\alpha$ . It is also obvious that  $F/p - \alpha \geq 0$  if and only if  $F - \alpha p \geq 0$ , for points at which  $p > 0$ . By [Lemma A.3](#) we have in local coordinates on some possibly smaller neighbourhood  $U$  of  $z$

$$F/p = \beta - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_{n-j(z)}^2 + (-1)^{\sigma(n-j(z)+1)}x_{n-j(z)+1} + \cdots + (-1)^{\sigma(n)}x_n. \quad (\text{A-2})$$

By further decreasing  $U$ , we can assume the following: if there are zero-level lines of  $F - \beta p$  emerging from  $z$  then  $U$  is bounded either by a smooth curve which crosses the zero-level lines of  $F - \beta p$  or, if  $z \in \partial\mathcal{D}$ , by the union of such a curve and a subarc of  $\partial\mathcal{D}$ . If there are no zero-level lines of  $F - \beta p$  emerging from  $z$ , then the zero set of  $F - \beta p$  in a  $\overline{\mathcal{D}}$ -neighbourhood of  $z$  is just  $\{z\}$ . We can then choose  $U$  to be such a neighbourhood, and we further assume that the boundary of  $U$  is either a circle or the union of an arc of a circle and a subarc of  $\partial\mathcal{D}$ . Let  $B$  be an even smaller neighbourhood of  $z$  such that  $\overline{B} \subset U$ .

First we consider the case that there are zero-level lines of  $F - \beta p$  emerging from  $z$ . Arguing in the same way as in the proof of [Proposition A.1](#) we can choose  $U$  and  $B$  in such a way that all components of  $\{F - \beta p \geq 0\} \cap (U \setminus B)$  are quadrilaterals such that their boundaries are made of two subarcs of  $\partial B$



**Figure 2.** Change of the excursion set near a quasicritical point of index 1 on a one-dimensional stratum. The analysis of other cases is very similar.

and  $\partial U$  and two arcs that are either subarcs of  $\{F - \beta p = 0\}$  or  $\partial \mathcal{D}$ . The flow (A-1) is uniformly bounded in a small neighbourhood of  $\{F - \beta p = 0\} \setminus B$  and there are no critical points in this neighbourhood. As before, this implies that there is a sufficiently small  $\delta > 0$  such that if we run the flow starting from  $\alpha = \beta$  and ending inside  $[\beta - \delta, \beta + \delta]$  then the curves  $\{F - \beta p = 0\} \setminus B$  move by less than  $\text{dist}(B, \mathcal{D} \setminus U)$ . This means that the topology of the excursion sets  $A_\alpha$  and nodal sets  $\{g_\alpha = 0\}$  outside of  $U$  does not change (as we pass from  $\alpha = \beta - \delta$  to  $\alpha = \beta + \delta$ ) and the number of arcs where the nodal set intersects  $\partial U$  does not change either. This proves that any change in the number of excursion set components which do not intersect  $\partial \mathcal{D}$  must happen inside of  $U$ .

In the case that there are no zero-level lines of  $F - \beta p$  emerging from  $z$ , we know that  $F - \beta p$  is bounded away from zero on  $U \setminus B$  and there are no critical points on this set. Therefore in this case we can also choose  $\delta > 0$  such that  $F - \alpha p$  is nonzero on  $U \setminus B$  for all  $\alpha \in [\beta - \delta, \beta + \delta]$ . Once again we conclude that any change in the topology of  $A_\alpha$  for this range of  $\alpha$  must occur inside  $U$ .

Inside of  $U$ , we consider the excursion sets  $\{F/p \geq \alpha\}$  rather than  $A_\alpha = \{F - \alpha p \geq 0\}$  because we can describe the former using the simple coordinates in (A-2). So depending on the dimension of the stratum containing  $z$  and the index of  $z$ , we have the following options for the *local behaviour* in  $U$  when  $\alpha$  is close to  $\beta$ :

- (1) Dimension 2 ( $z \in \mathcal{D}$ ): If the index  $\lambda$  is 0 or 2, then the nodal lines of  $F - \alpha p$  do not intersect  $\partial U$ . As  $\alpha$  moves from below  $\beta$  to above  $\beta$ , the component of  $\{F/p - \alpha \geq 0\}$  changes from a small disc, to a point  $z$ , to the empty set or the other way round. If the index is 1, then the excursion component in  $U$  changes from a single component bounded by a hyperbola, to two components separated by a hyperbola (or the other way round).
- (2) Dimension 1 ( $z \in \mathcal{E}$ ): At  $\alpha = \beta$  the excursion component is either above a parabola, in which case the topology does not change, or it is the domain between a subarc of  $\partial \mathcal{D}$  and a parabola. In the latter case, the excursion set in  $U$  can change from one component to two components (or the other way round). See Figure 2 for an example.

- (3) Dimension 0 ( $z \in \mathcal{C}$ ): In this case, if both  $\sigma(1)$  and  $\sigma(2)$  are the same, then  $\{z\}$  is an isolated component of the nodal set of  $F - \beta p$  and it can either disappear or become an interval. So the topology of the excursion set either stays the same or we add one component. If  $\sigma(1) \neq \sigma(2)$ , then the nodal line of  $F - \beta p$  is a line through  $z$ . When  $\alpha$  changes, this line shifts, but the number of excursion set components does not change.

Summing this up, we see that inside  $U$  the number of components of  $A_\alpha$  which do not intersect  $\partial\mathcal{D}$  cannot change by more than one, and outside  $U$  the number of such domains does not change at all. The arguments above also show that the same is true of the number of nodal components, and so this completes the proof.  $\square$

*Proof of Theorem 4.2.* This follows immediately from applying [Proposition A.4](#) to each quasicritical point of  $(F, p)$  and [Proposition A.1](#) at levels without such points.  $\square$

### Acknowledgements

Beliaev was partially funded by the Engineering and Physical Sciences Research Council (EPSRC) Fellowship EP/M002896/1 and partially supported by the Ministry of Science and Higher Education of Russia (subsidy in the form of a grant for creation and development of International Mathematical Centers, agreement no. 075-15-2019-1620, November 8, 2019). McAuley was supported by the European Research Council (ERC) Advanced Grant QFPROBA (grant number 741487). Muirhead was partially supported by the Australian Research Council (ARC) Discovery Early Career Researcher Award DE200101467.

The authors would like to thank the two referees for many comments which substantially improved the arrangement of our arguments and the first referee especially for pointing out an error in the original proof of [Lemma 3.11](#). The authors would also like to thank Benedetta Cavalli for making us aware of a mistake in the initial proof of [Theorem 2.15](#).

Beliaev is grateful for the hospitality of the Center for Advanced Studies at Skoltech, Moscow, where he was a visiting professor in the Fall of 2021.

### References

- [1] R. J. Adler and J. E. Taylor, *Random fields and geometry*, Springer, 2007. [MR](#) [Zbl](#)
- [2] K. S. Alexander, “Boundedness of level lines for two-dimensional random fields”, *Ann. Probab.* **24**:4 (1996), 1653–1674. [MR](#) [Zbl](#)
- [3] J.-M. Azaïs and M. Wschebor, *Level sets and extrema of random processes and fields*, Wiley, Hoboken, NJ, 2009. [MR](#) [Zbl](#)
- [4] J. Bardeen, J. Bond, N. Kaiser, and A. Szalay, “The statistics of peaks of Gaussian random fields”, *Astrophys. J.* **304** (1986), 15–61.
- [5] V. Beffara and D. Gayet, “Percolation of random nodal lines”, *Publ. Math. Inst. Hautes Études Sci.* **126** (2017), 131–176. [MR](#) [Zbl](#)
- [6] D. Beliaev, M. McAuley, and S. Muirhead, “On the number of excursion sets of planar Gaussian fields”, *Probab. Theory Related Fields* **178**:3–4 (2020), 655–698. [MR](#) [Zbl](#)
- [7] D. Beliaev, M. McAuley, and S. Muirhead, “Smoothness and monotonicity of the excursion set density of planar Gaussian fields”, *Electron. J. Probab.* **25** (2020), art. id. 93. [MR](#) [Zbl](#)

- [8] D. Beliaev, S. Muirhead, and A. Rivera, “A covariance formula for topological events of smooth Gaussian fields”, *Ann. Probab.* **48**:6 (2020), 2845–2893. [MR](#) [Zbl](#)
- [9] M. V. Berry, “Regular and irregular semiclassical wavefunctions”, *J. Phys. A* **10**:12 (1977), 2083–2091. [MR](#) [Zbl](#)
- [10] M. V. Berry, “Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature”, *J. Phys. A* **35**:13 (2002), 3025–3038. [MR](#) [Zbl](#)
- [11] E. Bogomolny and C. Schmit, “Percolation model for nodal domains of chaotic wave functions”, *Phys. Rev. Lett.* **88** (2002), art. id. 114102.
- [12] E. Bogomolny and C. Schmit, “Random wavefunctions and percolation”, *J. Phys. A* **40**:47 (2007), 14033–14043. [MR](#) [Zbl](#)
- [13] V. Cammarota and D. Marinucci, “A quantitative central limit theorem for the Euler–Poincaré characteristic of random spherical eigenfunctions”, *Ann. Probab.* **46**:6 (2018), 3188–3228. [MR](#) [Zbl](#)
- [14] V. Cammarota, D. Marinucci, and I. Wigman, “Fluctuations of the Euler–Poincaré characteristic for random spherical harmonics”, *Proc. Amer. Math. Soc.* **144**:11 (2016), 4759–4775. [MR](#) [Zbl](#)
- [15] S. Chatterjee, “A general method for lower bounds on fluctuations of random variables”, *Ann. Probab.* **47**:4 (2019), 2140–2171. [MR](#) [Zbl](#)
- [16] D. Cheng and A. Schwartzman, “Expected number and height distribution of critical points of smooth isotropic Gaussian random fields”, *Bernoulli* **24**:4B (2018), 3422–3446. [MR](#) [Zbl](#)
- [17] A. I. Elizarov, “On the variance of the number of stationary points of a homogeneous Gaussian field”, *Theory Probab. Appl.* **29**:3 (1985), 569–570. [Zbl](#)
- [18] A. Estrade and J. Fournier, “Number of critical points of a Gaussian random field: condition for a finite variance”, *Statist. Probab. Lett.* **118** (2016), 94–99. [MR](#) [Zbl](#)
- [19] A. Estrade and J. R. León, “A central limit theorem for the Euler characteristic of a Gaussian excursion set”, *Ann. Probab.* **44**:6 (2016), 3849–3878. [MR](#) [Zbl](#)
- [20] G. Grimmett, *Percolation*, 2nd ed., Grundle Math. Wissen. **321**, Springer, 1999. [MR](#) [Zbl](#)
- [21] D. G. C. Handron, “Generalized billiard paths and Morse theory for manifolds with corners”, *Topology Appl.* **126**:1-2 (2002), 83–118. [MR](#) [Zbl](#)
- [22] S. R. Jain and R. Samajdar, “Nodal portraits of quantum billiards: domains, lines, and statistics”, *Rev. Modern Phys.* **89**:4 (2017), art. id. 045005. [MR](#)
- [23] S. Janson, *Gaussian Hilbert spaces*, Cambridge Tracts in Mathematics **129**, Cambridge Univ. Press, 1997. [MR](#) [Zbl](#)
- [24] M. F. Kratz and J. R. León, “Hermite polynomial expansion for non-smooth functionals of stationary Gaussian processes: crossings and extremes”, *Stochastic Process. Appl.* **66**:2 (1997), 237–252. [MR](#) [Zbl](#)
- [25] M. Kratz and S. Vadlamani, “Central limit theorem for Lipschitz–Killing curvatures of excursion sets of Gaussian random fields”, *J. Theoret. Probab.* **31**:3 (2018), 1729–1758. [MR](#) [Zbl](#)
- [26] P. Kurlberg and I. Wigman, “Variation of the Nazarov–Sodin constant for random plane waves and arithmetic random waves”, *Adv. Math.* **330** (2018), 516–552. [MR](#) [Zbl](#)
- [27] N. Leonenko, *Limit theorems for random fields with singular spectrum*, Mathematics and its Applications **465**, Kluwer Academic, Dordrecht, 1999. [MR](#) [Zbl](#)
- [28] D. Marinucci and I. Wigman, “The defect variance of random spherical harmonics”, *J. Phys. A: Math. Theor.* **44**:35 (2011), art. id. 355206. [Zbl](#)
- [29] D. Marinucci and I. Wigman, “On the area of excursion sets of spherical Gaussian eigenfunctions”, *J. Math. Phys.* **52**:9 (2011), art. id. 093301. [MR](#) [Zbl](#)
- [30] D. Marinucci, M. Rossi, and I. Wigman, “The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics”, *Ann. Inst. Henri Poincaré Probab. Stat.* **56**:1 (2020), 374–390. [MR](#) [Zbl](#)
- [31] M. McAuley, *Excursion sets of planar gaussian fields*, Ph.D. thesis, University of Oxford, 2020, available at <https://www.proquest.com/docview/2467512615>. [Zbl](#)
- [32] S. Muirhead, “A second moment bound for critical points of planar Gaussian fields in shrinking height windows”, *Statist. Probab. Lett.* **160** (2020), art. id. 108698. [MR](#) [Zbl](#)



- [33] S. Muirhead and H. Vanneuille, “The sharp phase transition for level set percolation of smooth planar Gaussian fields”, *Ann. Inst. Henri Poincaré Probab. Stat.* **56**:2 (2020), 1358–1390. [MR](#) [Zbl](#)
- [34] D. Müller, “A central limit theorem for Lipschitz–Killing curvatures of Gaussian excursions”, *J. Math. Anal. Appl.* **452**:2 (2017), 1040–1081. [MR](#) [Zbl](#)
- [35] F. Nazarov and M. Sodin, “On the number of nodal domains of random spherical harmonics”, *Amer. J. Math.* **131**:5 (2009), 1337–1357. [MR](#) [Zbl](#)
- [36] F. Nazarov and M. Sodin, “Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions”, *J. Math. Phys. Anal. Geo.* **12**:3 (2016), 205–278. [MR](#) [Zbl](#)
- [37] F. Nazarov and M. Sodin, “Fluctuations in the number of nodal domains”, *J. Math. Phys.* **61**:12 (2020), art. id. 123302. [MR](#) [Zbl](#)
- [38] L. I. Nicolaescu, “A CLT concerning critical points of random functions on a Euclidean space”, *Stochastic Process. Appl.* **127**:10 (2017), 3412–3446. [MR](#) [Zbl](#)
- [39] I. Nourdin, G. Peccati, and M. Rossi, “Nodal statistics of planar random waves”, *Comm. Math. Phys.* **369**:1 (2019), 99–151. [MR](#) [Zbl](#)
- [40] P. Pranav, R. J. Adler, T. Buchert, H. Edelsbrunner, B. J. T. Jones, A. Schwartzman, H. Wagner, and R. van de Weygaert, “Unexpected topology of the temperature fluctuations in the cosmic microwave background”, *A&A* **627** (2019), art. id. A163.
- [41] L. Priya, “Concentration for nodal component count of Gaussian Laplace eigenfunctions”, preprint, 2020. [arXiv 2012.10302](#)
- [42] A. Rivera, “Talagrand’s inequality in planar Gaussian field percolation”, *Electron. J. Probab.* **26** (2021), art. id. 26. [MR](#) [Zbl](#)
- [43] A. Rivera and H. Vanneuille, “Quasi-independence for nodal lines”, *Ann. Inst. Henri Poincaré Probab. Stat.* **55**:3 (2019), 1679–1711. [MR](#) [Zbl](#)
- [44] A. Rivera and H. Vanneuille, “The critical threshold for Bargmann–Fock percolation”, *Ann. H. Lebesgue* **3** (2020), 169–215. [MR](#) [Zbl](#)
- [45] E. V. Slud, “MWI representation of the number of curve-crossings by a differentiable Gaussian process, with applications”, *Ann. Probab.* **22**:3 (1994), 1355–1380. [MR](#) [Zbl](#)
- [46] S. A. Vakhrameev, “Morse lemmas for smooth functions on manifolds with corners”, *J. Math. Sci.* **100**:4 (2000), 2428–2445. [MR](#) [Zbl](#)
- [47] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, 1944. [MR](#) [Zbl](#)
- [48] I. Wigman, “Fluctuations of the nodal length of random spherical harmonics”, *Comm. Math. Phys.* **298**:3 (2010), 787–831. [MR](#) [Zbl](#)
- [49] I. Wigman, “On the nodal lines of random and deterministic Laplace eigenfunctions”, pp. 285–297 in *Spectral geometry*, edited by A. H. Barnett et al., Proc. Sympos. Pure Math. **84**, Amer. Math. Soc., Providence, RI, 2012. [MR](#) [Zbl](#)
- [50] K. J. Worsley, S. Marrett, P. Neelin, A. C. Vandal, K. J. Friston, and A. C. Evans, “A unified statistical approach for determining significant signals in images of cerebral activation”, *Hum. Brain Mapp.* **4**:1 (1996), 58–73.

Received 2 Dec 2020. Revised 20 Aug 2021. Accepted 15 Sep 2021.

DMITRY BELIAEV: [belyaev@maths.ox.ac.uk](mailto:belyaev@maths.ox.ac.uk)

Mathematical Institute, University of Oxford, United Kingdom

MICHAEL MCAULEY: [michael.mcauley@helsinki.fi](mailto:michael.mcauley@helsinki.fi)

Mathematical Institute, University of Oxford, United Kingdom

Current address: Department of Mathematics and Statistics, University of Helsinki, Finland

STEPHEN MUIRHEAD: [smui@unimelb.edu.au](mailto:smui@unimelb.edu.au)

School of Mathematical Sciences, Queen Mary University of London, United Kingdom

Current address: School of Mathematics and Statistics, University of Melbourne, Australia

**EDITORS-IN-CHIEF**

Alexei Borodin Massachusetts Institute of Technology, United States  
Hugo Duminil-Copin IHÉS, France & Université de Genève, Switzerland  
Robert Seiringer Institute of Science and Technology, Austria  
Sylvia Serfaty Courant Institute, New York University, United States

**EDITORIAL BOARD**

Nalini Anantharaman Université Paris-Sud XI, France  
Scott Armstrong Courant Institute, New York University, United States  
Roland Bauerschmidt University of Cambridge, UK  
Ivan Corwin Columbia University, United States  
Mihalis Dafermos Princeton University, United States  
Semyon Dyatlov MIT, United States  
László Erdős IST Austria  
Yan Fyodorov King's College London, UK  
Christophe Garban Université Claude Bernard Lyon 1, France  
Alessandro Giuliani Università degli studi Roma Tre, Italy  
Pierre-Emmanuel Jabin Pennsylvania State University, United States  
Mathieu Lewin Université Paris Dauphine & CNRS, France  
Eyal Lubetzky Courant Institute, New York University, United States  
Jean-Christophe Mourrat École normale supérieure de Lyon & CNRS, France  
Laure Saint Raymond École normale supérieure de Lyon & CNRS, France  
Benjamin Schlein Universität Zürich, Switzerland  
Vlad Vicol Courant Institute, New York University, United States  
Simone Warzel Technische Universität München, Germany

**PRODUCTION**

Silvio Levy (Scientific Editor)  
[production@msp.org](mailto:production@msp.org)

---

See inside back cover or [msp.org/pmp](https://msp.org/pmp) for submission instructions.

Probability and Mathematical Physics (ISSN 2690-1005 electronic, 2690-0998 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

PMP peer review and production are managed by EditFlow<sup>®</sup> from MSP.

PUBLISHED BY  
 **mathematical sciences publishers**  
nonprofit scientific publishing  
<https://msp.org/>

© 2022 Mathematical Sciences Publishers

# PROBABILITY and MATHEMATICAL PHYSICS

3:1

2022

Gaussian free fields and Riemannian rigidity	1
Nguyen Viet Dang	
Large-time behavior of small-data solutions to the Vlasov–Navier–Stokes system on the whole space	35
Daniel Han-Kwan	
Mathematics of magic angles in a model of twisted bilayer graphene	69
Simon Becker, Mark Embree, Jens Wittsten and Maciej Zworski	
Fluctuations of the number of excursion sets of planar Gaussian fields	105
Dmitry Beliaev, Michael McAuley and Stephen Muirhead	
Joint moments of a characteristic polynomial and its derivative for the circular $\beta$ -ensemble	145
Peter J. Forrester	
Global solutions to the supercooled Stefan problem with blow-ups: regularity and uniqueness	171
François Delarue, Sergey Nadtochiy and Mykhaylo Shkolnikov	