Topology of smooth Gaussian fields

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Slides available at
`https://michael-mcauley.github.io`

Outline

1. Motivation

2. Wiener chaos method for local functionals

3. Methods for non-local functionals

Smooth Gaussian fields

- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a stationary, mean-zero C^2 Gaussian field.
- The distribution of f is specified by its covariance function,

$$K(x - y) := \mathrm{Cov}[f(x), f(y)].$$

- We will consider the geometry/topology of the excursion sets,

$$\{f \geq \ell\} := \left\{x \in \mathbb{R}^d \;\middle|\; f(x) \geq \ell\right\} \qquad \text{for } \ell \in \mathbb{R}.$$

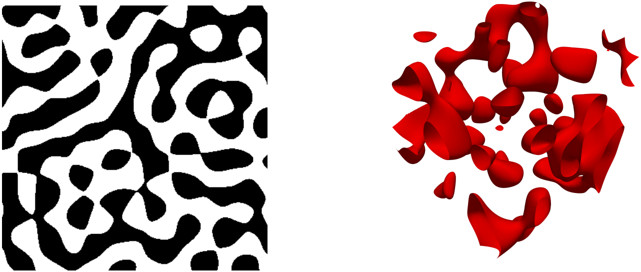

Figure: Excursion sets of Gaussian fields in two and three dimensions.

Berry's conjecture [7]: on generic d-dimensional Riemannian manifolds, high-frequency eigenfunctions of the Laplacian can be locally approximated by the Gaussian field with

$$K(x) = J_{\frac{d-2}{2}}(|x|)|x|^{-\frac{d-2}{2}} \sim \cos(|x| - c_d)|x|^{-\frac{d-1}{2}} \qquad \text{as } |x| \to \infty,$$

known as monochromatic random waves.

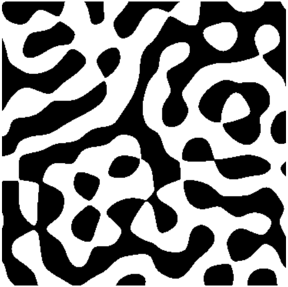

Figure: Excursion set of planar monochromatic random waves.

Motivation

1) Studying classes of functions

Hilbert's 16th problem concerns the zero set of homogeneous polynomials. There is a canonical Gaussian measure on such polynomials which behaves locally like the stationary field with

$$K(x) = \exp(-|x|^2/2),$$

known as the Bargmann-Fock field [14].

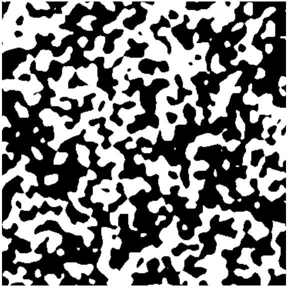

Figure: Excursion set of the planar Bargmann-Fock field.

Motivation

2) Percolation theory

- Percolation theory studies the long-range connectivity properties of random models.
- **Bernoulli percolation**: adjacent points of $\mathbb{Z}^2$ are joined by an edge independently with probability p.

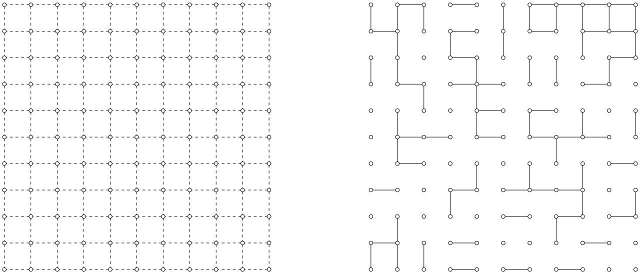

Figure: A section of the square lattice $\mathbb{Z}^2$ (left) and a realisation of the Bernoulli percolation model with $p = 0.4$ on this section (right).

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Motivation

2) Percolation theory

- Progress has been made recently in studying percolation of Gaussian excursion sets [2].
- **Phase transition**: for a given field, there is a critical level ℓ_c such that
  - for $\ell > \ell_c$, $\{f \geq \ell\}$ contains only bounded components,
  - for $\ell < \ell_c$, $\{f \geq \ell\}$ contains a unique unbounded component.

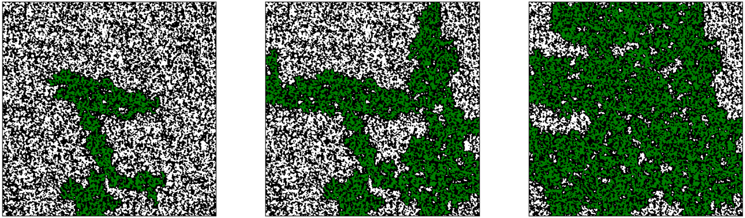

Figure: Excursion sets $\{f \geq \ell\}$ for $\ell = 0.05$ (left), $\ell = 0$ (middle) and $\ell = -0.05$ (right). Largest component highlighted in green.

Motivation

3) Statistical applications

- Gaussian fields arise in many areas of science:
  - Medical imaging [25],
  - Cosmology [22],
  - Topological data analysis [1].
- Geometric/topological properties of excursion sets can be used as test statistics. (See [24] for an overview.)

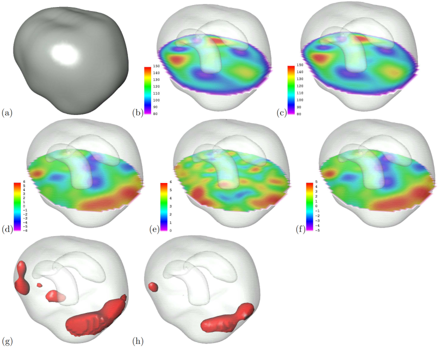

Figure: Measurements from a PET study of brain activity during a reading task. (Source: [24]).

Questions of interest

- What are the geometric and topological properties of smooth Gaussian excursion sets?
- Focus on **'approximately additive'** functionals:

Geometric functionals
- Volume
- Boundary volume

Topological functionals
- Number of connected components
- Betti numbers
- Euler characteristic

- What is the expectation, variance and distribution of such functionals on a bounded domain?
- How does this depend on the size of the domain? the level of the excursion set? the covariance of the field?

Local vs non-local functionals

- A functional of a random field is **local** if it can be expressed as:

$$\int_D \varphi(f(x), \nabla f(x), \nabla^2 f(x))\, \mu(dx)$$

- Examples

Local functionals

- Volume $\int_D \mathbb{1}_{f(x)\geq \ell}\, dx$
- Boundary volume $\int_D \mathbb{1}_{f(x)=\ell} \mathcal{H}^{d-1}(dx)$
- Number of critical points
- Euler characteristic

Non-local functionals

- Number of connected components
- Betti numbers
- Volume of the unbounded component

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Outline

Let $L_R := \mathcal{H}^{d-1}(\{f = \ell\} \cap [-R/2, R/2]^d)$ denote the boundary measure of excursion sets.

Theorem (Kratz-Vadlamani [13])

For the Bargmann-Fock field, there exist explicit $c_\ell, C_\ell > 0$ such that

1. *For all $R > 0$, $\mathbb{E}[L_R] = c_\ell R^d$*
2. *As $R \to \infty$,*

$$\frac{\mathrm{Var}[L_R]}{R^d} \to C_\ell \qquad \textit{and} \qquad \frac{L_R - \mathbb{E}[L_R]}{\sqrt{\mathrm{Var}[L_R]}} \xrightarrow{d} \mathcal{N}(0,1).$$

- Analogous results hold for other local functionals and fields with fast correlation decay.

Theorem (Wigman, Nourdin, Peccati, Rossi, Marinucci)

For planar monochromatic random waves, there exist $c_\ell, C_\ell > 0$ such that

1. $\mathbb{E}[L_R] = c_\ell R^2$
2. *As* $R \to \infty$

$$\mathrm{Var}[L_R] \sim \begin{cases} C_\ell R^3 & \text{if } \ell \neq 0 \\ C_0 R^2 \log(R) & \text{if } \ell = 0. \end{cases} \qquad \text{and} \qquad \frac{L_R - \mathbb{E}[L_R]}{\sqrt{\mathrm{Var}[L_R]}} \xrightarrow{d} \mathcal{N}(0,1).$$

- The anomalous behaviour at $\ell = 0$ is known as Berry cancellation.
- Analogous results are known for other functionals and in higher dimensions.
- Non-Gaussian distributional limits are also possible.

Wiener chaos expansion

Let $\mathcal{G}$ be a set of centred jointly Gaussian variables. Let $\mathcal{P}_n$ be the space of all polynomials of degree $\leq n$ in $\mathcal{G}$.
The n-th **Wiener chaos** of $\mathcal{G}$ is $\mathcal{G}^{:n:} := \overline{\mathcal{P}_n} \cap \overline{\mathcal{P}_{n-1}}^{\perp}$.

Theorem

Let the random variable F be square integrable and $\sigma(\mathcal{G})$-measurable, then

$$F \stackrel{L^2}{=} \sum_{n=0}^{\infty} Q_n[F]$$

where Q_n denotes projection onto $\mathcal{G}^{:n:}$.

Remark: For local functionals we can compute chaos projections <u>pointwise</u>.

Hermite expansion

- Consider
$$\Phi_R := \int_{[-R,R]^d} \varphi(f(x) - \ell)\, dx.$$
- The Hermite polynomials $(H_n)_{n\geq 0}$ can be defined inductively by setting
$$H_0(x) = 1 \quad \text{and} \quad H_{n+1}(x) = xH_n(x) - H_n'(x).$$
They form an orthogonal (Schauder) basis for $L^2(e^{-x^2/2}\, dx)$.
- By the expansion $\varphi(\cdot - \ell) = \sum_n a_n(\ell) H_n$,
$$Q_n[\Phi_R] = a_n(\ell) \int_{[-R,R]^d} H_n(f(x))\, dx$$
- By Fubini and a property of Hermite polynomials
$$\begin{aligned}\mathrm{Var}[Q_n[F_R]] &= a_n(\ell)^2 \iint_{[-R,R]^{2d}} \mathrm{Cov}[H_n(f(x)), H_n(f(y))]\, dxdy \\ &= a_n(\ell)^2 n! \iint_{[-R,R]^{2d}} K(x-y)^n\, dxdy\end{aligned}$$

Chaos expansion for a local functional

Recall

$$\mathrm{Var}[Q_n[\Phi_R]] = a_n(\ell)^2 n! \int\int_{[-R,R]^{2d}} K(x-y)^n \, dxdy.$$

- For the Bargmann-Fock field $K(x) = \exp(-|x|^2/2)$, so that

  $$\mathrm{Var}[Q_n[\Phi_R]] \sim a_n(\ell)^2 C_{K,n} R^d$$

  and all chaoses contribute in the limit.
- For planar monochromatic random waves $K(x) \sim \cos(|x|-c)|x|^{-1/2}$ so that

  $$\mathrm{Var}[Q_n[\Phi_R]] \sim C_{K,n} \times \begin{cases} a_2(\ell)^2 R^3 & \text{if } n=2 \\ a_4(\ell)^2 R^2 \log(R) & \text{if } n=4 \\ a_n(\ell)^2 O(R^2) & \text{otherwise.} \end{cases}$$

  and a single chaos typically dominates.
- Distributional limits can be studied using the method of moments or fourth-moment theorem.

Local functionals

Summary and references

- Method generalises to
  - Local functionals $\Phi(D) = \int_D \varphi(f(x), \nabla f(x), \nabla^2 f(x))\, \mu(dx)$
  - Fields on manifolds
  - Quantitative CLTs
  - Correlations between functionals at different levels

  although in practice the computations can be infeasible.
- Useful references/significant results:
  - Breuer-Major and Dobrushin-Major theorems [8, 9]
  - Fourth moment theorem via Malliavin-Stein [21]
  - CLT for the Euler characteristic [10, 13]
  - Survey of the literature [23]

Outline

1. Motivation

2. Wiener chaos method for local functionals

3. Methods for non-local functionals

Non-local (additive) functionals

What is known?

No unifying theory, but many partial results:

Result	Method
Concentration bounds [20]	Gaussian isoperimetric inequality
Law of large numbers [18]	Ergodic argument
Variance bounds [19, 5, 4]	Coupling, interpolation
Central limit theorem [3, 16, 12]	Martingale techniques

Open problems:

- Stronger results (e.g., quantitative CLTs)
- Proving Berry cancellation
- Framework to handle different covariance structures

- A set $\mathcal{X}$ of random variables is associated if for any $X_1, \ldots, X_n \in \mathcal{X}$ and bounded, non-decreasing F, G, $\mathrm{Cov}[F(X_1, \ldots, X_n), G(X_1, \ldots, X_n)] \geq 0$.
- **Newman's inequality:** For associated $X = (X_1, \ldots, X_n)$ and Lipschitz continuous F, G,

$$|\mathrm{Cov}[F(X), G(X)]| \leq C \|F\|_{\mathrm{Lip}} \|G\|_{\mathrm{Lip}} \sum_{i,j=1}^{n} |\mathrm{Cov}[X_i, X_j]|.$$

  This property is known as quasi-association.
- Many limit theorems hold for block sums of stationary, quasi-associated random fields $(X_j)_{j \in \mathbb{Z}^d}$ which satisfy

$$\sum_{j \in \mathbb{Z}^d} |\mathrm{Cov}[X_0, X_j]| < \infty$$

Proposition

Let $X, \widetilde{X} \sim \mathcal{N}(0, K)$ be independent, n-dimensional and $X^t = tX + \sqrt{1-t^2}\widetilde{X}$. If $F, G : \mathbb{R}^n \to \mathbb{R}$ are Lipschitz continuous, then

$$\mathrm{Cov}[F(X), G(X)] = \sum_{i,j} K_{i,j} \int_0^1 \mathbb{E}[\partial_i F(X) \partial_j G(X^t)]\, dt.$$

Proof.

Since $X^1 = X$ and $X^0 = \widetilde{X}$

$$\begin{aligned}\mathrm{Cov}[F(X), G(X)] &= \mathbb{E}[F(X)G(X^1)] - \mathbb{E}[F(X)G(X^0)] \\ &= \int_0^1 \frac{d}{dt}\mathbb{E}[F(X)G(X^t)]\, dt.\end{aligned}$$

The result follows from 1) differentiating through the integral, 2) Gaussian IBP and 3) standard properties of Gaussian conditioning. □

Beliaev-Muirhead-Rivera [6] established an analogous result for topological events:

Theorem

Let $f, \widetilde{f}$ be independent centred Gaussian fields with covariance function K and $f^t = tf + \sqrt{1-t^2}\widetilde{f}$. For compact $D_1, D_2 \subset \mathbb{R}^d$, let N_i denote the number of components of $\{f \geq \ell\} \cap D_i$. For Lipschitz $F, G : \mathbb{R} \to \mathbb{R}$

$$\mathrm{Cov}[F(N_1), G(N_2)] = \int_{D_1 \times D_2} K(x-y) \int_0^1 \mu^t_{x,y}(d_x F(N_1) d_y G(N_2^t))\, dt dx dy$$

where $\mu^t_{x,y}$ denotes a 'pivotal measure' which conditions f and f^t to have critical points at x and y at level ℓ, and d_x, d_y denote the change under a positive perturbation at x and y.

Corollary

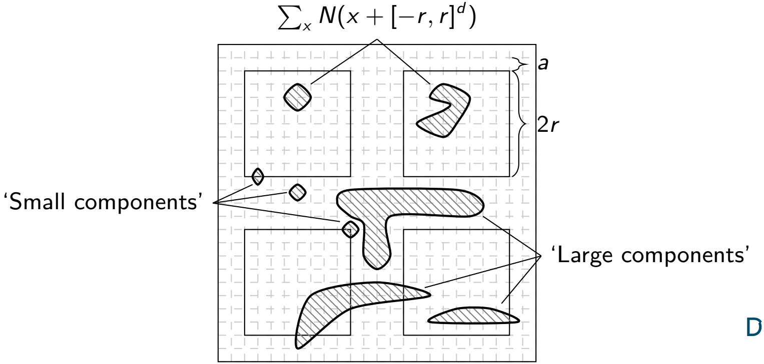
For 'nice' domains D_1, D_2

$$|\mathrm{Cov}[F(N_1), G(N_2)]| \leq C \|F\|_{\mathrm{Lip}} \|G\|_{\mathrm{Lip}} \int_{D_1 \times D_2} |K(x-y)|\, dx dy.$$

Proposition

Let $N(D)$ denote the number of components of $\{f \geq \ell\}$ in D. For $1 \ll a \ll r \ll R$ we divide $[-R,R]^d$ into boxes of side-length $2r$ separated at scale a. Let χ_R denote the centres of the r-boxes, then

$$R^{-d}\mathrm{Var}\Big[N([-R,R]^d) - \sum_{x\in\chi_R} N(x+[-r,r]^d)\Big] \to 0$$



Emulating the proofs for discrete quasi-associated fields, we have:

Theorem (M. 2026)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be the Bargmann-Fock field and $\ell \in \mathbb{R}$, then as $R \to \infty$

$$\frac{\mathrm{Var}[N_R]}{R^d} \to \sigma_\ell^2 > 0 \qquad \textit{and} \qquad \frac{N_R - \mathbb{E}[N_R]}{R^{d/2}} \xrightarrow{d} \mathcal{N}(0, \sigma_\ell^2).$$

Moreover for $|\ell| > \ell_c$ (the percolation threshold)

$$d_{\mathrm{Kol}}\left(\frac{N_R - \mathbb{E}[N_R]}{R^{d/2}}, \mathcal{N}(0, \sigma_\ell^2)\right) \leq CR^{-\frac{d}{2}\frac{1}{2d+1}} \log(R)$$

Remark:

- The result applies to general fields with integrable covariance function.
- Quasi-association yields further results including a law of large numbers, concentration estimates and the law of the iterated logarithm.

Method 2: Wiener chaos expansion

Result

- Let $f : \mathbb{Z}^d \to \mathbb{R}$ be the Gaussian free field (in $d \geq 3$), so that

$$K(x-y) \sim c_d |x-y|^{-(d-2)}.$$

- Let N_R be the number of connected components in $\mathbb{Z}^d$ of $\{f \geq \ell\} \cap [-R, R]^d$ and define the mean clusters-per-vertex as

$$\mu(\ell) := \lim_{R\to\infty} \frac{\mathbb{E}[N_R]}{(2R)^d}.$$

Theorem (M.-Muirhead 2025 [17])

For $\ell \neq \ell_c$ (the percolation threshold),

$$\mathrm{Var}[N_R] \sim c_\ell \times \begin{cases} R^5 & \text{if } \mu'(\ell) \neq 0 \\ R^4 & \text{if } \mu'(\ell) = 0, \mu''(\ell) \neq 0 \\ R^3 \log R & \text{if } \mu'(\ell) = \mu''(\ell) = 0, \mu'''(\ell) \neq 0 \\ R^3 & \text{otherwise.} \end{cases}$$

In the second case, the (normalised) limiting distribution is a Hermite distribution, in all other cases it is Gaussian.

Proposition

Let $D \subset \mathbb{Z}^d$ be finite and $\Phi \in C^\infty(\mathbb{R}^D)$, then

$$Q_n[\Phi(f)] = \frac{1}{n!} \sum_{x_1,\dots,x_n \in D} \mathbb{E}[\partial_{x_1} \dots \partial_{x_n} \Phi(f)] \, {:}f(x_1)\dots f(x_n){:}$$

where the Wick polynomial ${:}f(x_1)\dots f(x_n){:}$ is defined as $Q_n[f(x_1)\dots f(x_n)]$.

- Proof 1: Elementary argument using Gaussian integration by parts.
- Proof 2: Stroock formula:

$$\begin{aligned} Q_n[\Phi(f)] &= I_n\left(\frac{1}{n!}\mathbb{E}[D^n\Phi(f)]\right) \\ &= \frac{1}{n!} \sum_{x_1,\dots,x_n \in D} \mathbb{E}[\partial_{x_1} \dots \partial_{x_n} \Phi(f)] I_n(e_1 \otimes \dots \otimes e_n). \end{aligned}$$

Proposition

For $R \geq 1$, denoting $\Lambda_R = [-R, R]^d \cap \mathbb{Z}^d$,

$$Q_n[N_R(f)] = \frac{1}{n!} \sum_{x_1,\dots,x_n \in \Lambda_R} P_R(x_1, \dots, x_n) :f(x_1) \dots f(x_n):$$

where the pivotal intensity P_R is defined for distinct points $\underline{x} = (x_1, \dots, x_n)$ as

$$P_R(\underline{x}) = \mathbb{E}[d_{x_1} \dots d_{x_n} N_R(f) | f(\underline{x}) = \ell] \varphi_{f(\underline{x})}(\ell),$$

$\varphi_{f(\underline{x})}$ is the density of $f(\underline{x})$, and d_x is the discrete derivative at x.

Open questions:

- Chaos expansion for smooth fields?
- Cancellations for monochromatic random waves?
- What happens at the critical level?

Useful references:

- Covariance formula for topological events [6].
- Chaos expansion for non-local functionals [17, 11].
- Quasi-association for non-local functionals [15].

Thank you for listening!

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