Topology of smooth Gaussian fields

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Slides available at
https://michael-mcauley.github.io

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Outline

1. Motivation
2. Percolation of excursion sets
3. Quasi-additive functionals

Smooth Gaussian fields

- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a stationary, mean-zero C^2 Gaussian field.
- The distribution of f is specified by its covariance function,

$$K(x - y) := \mathrm{Cov}[f(x), f(y)].$$

- We will consider the geometry/topology of the excursion sets,

$$\{f \geq \ell\} := \left\{x \in \mathbb{R}^d \;\middle|\; f(x) \geq \ell\right\} \qquad \text{for } \ell \in \mathbb{R}.$$

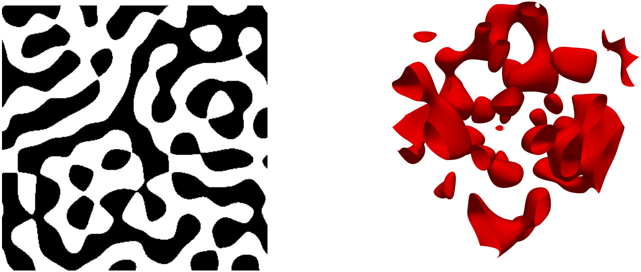

Figure: Excursion sets of Gaussian fields in two and three dimensions.

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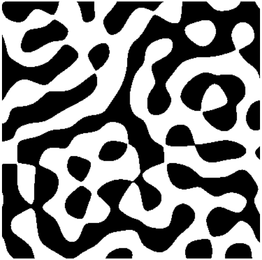
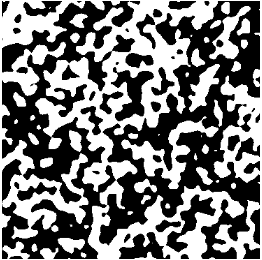
1. **Quantum chaos:** on generic manifolds, high-frequency eigenfunctions of the Laplacian can be approximated by monochromatic random waves (Berry's conjecture) [Ber77].
2. **Algebraic geometry:** Hilbert's 16th problem concerns the zero set of homogeneous polynomials, which can be described by the Bargmann-Fock field [LL14].



Monochromatic random waves
$K(x) = J_0(|x|)$.



Bargmann-Fock field
$K(x) = \exp(-|x|^2/2)$.

Motivation

2) Statistical applications

- Gaussian fields arise in many areas of science:
  - Medical imaging [Wor+96],
  - Cosmology [Pra+19],
  - Topological data analysis [Adl+10].
- Geometric/topological properties of excursion sets can be used as test statistics. (See [Wor96] for an overview.)

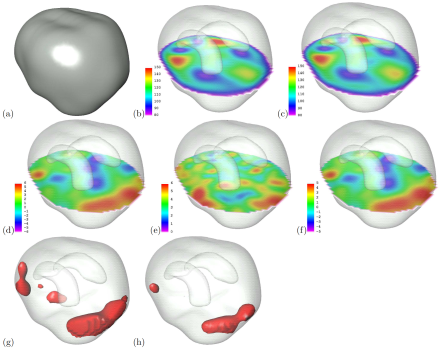


Figure: Measurements from a PET study of brain activity during a reading task. (Source: [Wor96]).

Geometry vs topology of Gaussian fields

Geometry:

- A classical topic with well developed theory [AT07; Wig24].
- Many geometric functionals take the form

$$\Phi(D, f) = \int_D \varphi(f(x), \nabla f(x), \nabla^2 f(x) \, \mu(dx),$$

  e.g. volume of excursion sets $\int_D \mathbb{1}_{f(x)\geq \ell} \, dx$.
- Can be analysed as 'sums' over growing D.

Topology:

- A relatively new topic with significant progress but many open questions [Bel23].
- Variety of methods used: no unified framework.

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Gaussian field percolation

Rough question: when do Gaussian excursion sets have long-range connections?

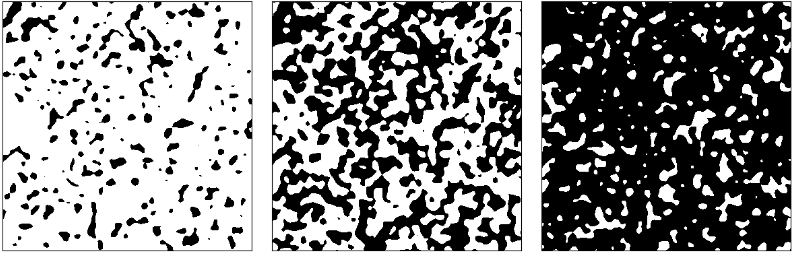


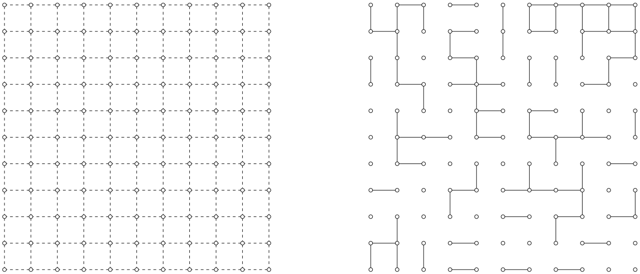
Figure: $\{f \geq 1\}$, $\{f \geq 0\}$ and $\{f \geq -1\}$ in black for Bargmann-Fock.

Precise questions:

- When does $\{f \geq \ell\}$ contain an unbounded component?
- Is the unbounded component unique?
- What is the probability of a component with diameter $\geq n$?

Benchmark: discrete percolation

- **Bernoulli percolation**: adjacent points of $\mathbb{Z}^d$ are joined by an edge independently with probability p.



- Inspires results and (some) techniques for Gaussian percolation.

Covariance function classes

Three general classes of covariance function are considered in the literature:

Case 1: K is **integrable**

- Example: Bargmann-Fock field

$$K(x) = \exp\left(-|x|^2/2\right).$$

Case 2: K is **regularly varying** at infinity with index $\alpha \in (0, d)$

- Example: Cauchy field

$$K(x) = (1 + |x|^2)^{-\alpha/2}.$$

Case 3: K is **oscillating and slowly decaying**

- Example: two-dimensional monochromatic random waves

$$K(x) = J_0(|x|) \sim \sqrt{\frac{2}{\pi}} \cos(|x| - \pi/4)|x|^{-1/2} \quad \text{as } |x| \to \infty.$$

Gaussian field percolation

Results

- **Phase transition**: there is a critical level ℓ_c such that with probability 1,
  - for $\ell > \ell_c$, $\{f \geq \ell\}$ contains only bounded components,
  - for $\ell < \ell_c$, $\{f \geq \ell\}$ contains an unbounded component.

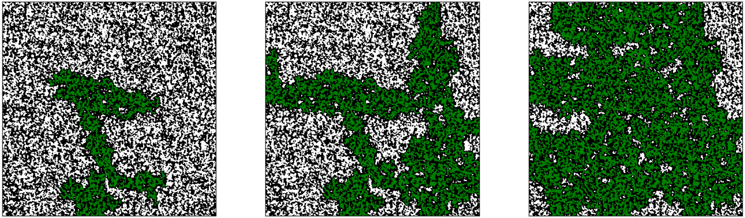


Figure: $\{f \geq 0.05\}$, $\{f \geq 0\}$ and $\{f \geq -0.05\}$. Largest component in green.

- Cases 1 & 2:
  - Phase transition holds for $d \geq 2$ [BG17; RV20; Sev22; Mui24].
  - Unbounded component is unique [Sev24].
  - In $d = 2$, no unbounded component at $\ell_c = 0$ [MV20].
- Case 3:
  - Phase transition holds for $d = 2$ [Mui+23].

Sharp phase transition:

- (Subcritical) For $\ell > \ell_c$,

$$\mathbb{P}\left(0 \xleftrightarrow{\{f \geq \ell\}} \partial B_n\right) \leq r_n \to 0.$$

- (Supercritical) For $\ell < \ell_c$,

$$\mathbb{P}\left(0 \xleftrightarrow{\{f \geq \ell\}<\infty} \partial B_n\right) \leq r_n \to 0.$$

- Known results: [MV20; Sev22; Mui24; Mui+23]

	Case 1 (Integrable)	Case 2 (Regularly varying)	Case 3 (Oscillating)
Subcritical	✓	✓	✘
Supercritical	✘	✗	✗

Open problems: Gaussian field percolation

- Monochromatic random waves (Case 3):
  - Phase transition for $d > 2$
  - (No) percolation at $\ell_c = 0$ for $d = 2$
- Supercritical sharpness
- Scaling limits of Gaussian level lines (SLE?)

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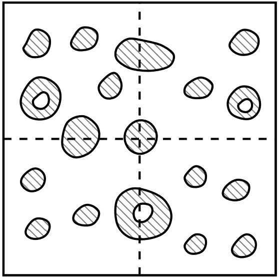
Quasi-additive functionals

- Quasi-additive functionals are those which are ‘approximately additive’ over domains.
- Examples:
  - Number of connected components
  - Betti numbers
  - Volume of unbounded component
- Questions:
  - Statistics on large domains?
  - Dependence on field and level?



- **Benchmark:** Geometric functionals
  - Expect similar behaviour
  - Different techniques required

Law of large numbers

Let N_R denote the number of components of $\{f \geq \ell\}$ in $[-R/2, R/2]^d$.

Theorem (Nazarov-Sodin 2016)

If f is ergodic, then there exists $\mu(\ell) \geq 0$ such that

$$\lim_{R\to\infty} \frac{N_R}{R^d} = \mu(\ell),$$

almost surely and in L^1.

Remark

- *Very weak assumptions required.*
- *Proof uses ergodicity and a double-scaling argument.*

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Theorem (Beliaev-M.-Muirhead 2024)

Let K be integrable and $\ell \in \mathbb{R}$, then as $R \to \infty$

$$\frac{\mathrm{Var}[N_R]}{R^d} \to \sigma_\ell^2 > 0 \qquad \text{and} \qquad \frac{N_R - \mathbb{E}[N_R]}{R^{d/2}} \xrightarrow{d} \mathcal{N}(0, \sigma_\ell^2).$$

Theorem (M. 2026)

N_R satisfies an almost sure and (for $|\ell| > \ell_c$) quantitative CLT as $R \to \infty$

- Let $f : \mathbb{Z}^d \to \mathbb{R}$ be the Gaussian free field (in $d \geq 3$), so that $K(x) \sim c_d |x|^{-(d-2)}$.

Theorem (M.-Muirhead 2025)

For $d = 3$ and $\ell \neq \ell_c$,

$$\mathrm{Var}[N_R] \sim c_\ell \times \begin{cases} R^5 & \text{if } \mathcal{R}(\ell) = 1 \\ R^4 & \text{if } \mathcal{R}(\ell) = 2 \\ R^3 \log R & \text{if } \mathcal{R}(\ell) = 3 \\ R^3 & \text{if } \mathcal{R}(\ell) \geq 4, \end{cases}$$

where $\mathcal{R} : \mathbb{R} \to \mathbb{N}$ is the 'effective Hermite rank' of N_R.

In the second case, the (normalised) limiting distribution is a Hermite distribution, in all other cases it is Gaussian.

Theorem (Beliaev-M.-Muirhead 2022, 2025)

Let f be monochromatic random waves, then

- *For all $\ell \in \mathbb{R}$*

$$\mathrm{Var}[N_R] \leq CR^{(3d+1)/2}.$$

- *For $d = 2$, and $\ell \neq 0$ such that $\mu'(\ell) \neq 0$*

$$\mathrm{Var}[N_R] \geq cR^3.$$

Remark

- *Expect that $\mathrm{Var}[N_R] \sim R^{d+1}$ for generic ℓ.*
- *Berry cancellation likely to occur at special levels.*

Open problems: quasi-additive functionals

- Monochromatic random waves:
  - Variance order?
  - Limiting distribution?
  - Berry cancellation?
- Unified framework for geometric and topological functionals.
- An exotic distribution at the critical level ℓ_c?

Thank you for listening!

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